THE

MATHEMATICAL GAZETTE.

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WITH THE CO-OPERATION OF
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ASSOCIATION FOR THE REFORM OF GEOMETRICAL TEACHING.*

CONFERENCE, JANUARY 17TH, 1871.

The Association will first be organised.

The following Resolutions will then be proposed—

(1) That the main object of this Association is to induce all Conductors of Examinations, at which pupils who have been trained under different systems present themselves, to frame their questions independently of any particular text book; and that with a view to this object, the members present at this meeting do pledge themselves to use every effort to increase the numbers, and extend the influence of the Association.

(2) That with a further view of extending the influence of the Association, Local Secretaries be appointed for different parts of the kingdom, whose office it shall be to collect information, to make the objects of the Association more generally known in their immediate neighbourhood, and to communicate on all matters of interest with the Central Committee.

(3) That the Local Secretaries, *ipso facto*, be Members of the Committee of Management.

(4) That all Members of the Association shall collect information with regard to text books and methods of teaching Geometry in England and other countries, and that such information shall be forwarded to any Secretary or Local Secretary of the Association.

*We are indebted to the kindness of the Rev. W. D. Bushell, M.A., elected Member of the Association at its formation in 1871, for the opportunity of reprinting the first summons to a conference for the organisation of the A.I.G.T.

(5) That the Committee of Management shall, from time to time, print and circulate among members and others, such information as they may consider valuable.

(6) That this meeting is of opinion that in any new text book—

(a) the following principles, only partially or not at all recognised by Euclid, should be adopted :—

(i) Hypothetical constructions.

(ii) The arithmetical definition of proportion.

(iii) Superposition.

(iv) The conception of a moving point, and of a revolving line.

(b) the following limitations should be removed :—

(i) The restriction of the number of axioms to those only which admit of no proof.

(ii) The restriction which excludes all angles not less than two right angles.

(c) modern terms, such as locus, projection, &c., should be introduced.

These points will be voted upon in detail.

PRESIDENTIAL ADDRESS TO MATHEMATICAL ASSOCIATION.

JAN. 11, 1911.

BY PROF. H. H. TURNER, D.Sc., D.C.L., F.R.S.

IN selecting a topic for his annual address, your President has a wealth of choice sufficient to be an embarrassment. On the one hand there are the varied activities of the Association itself; and though it may need courage and skill to deal with the delicate questions of the present moment, there is less need of exceptional virtues in reviewing the achievements of the past, or anticipating the enterprises of the future. On the other hand there are the doings in the world of Science outside the Association, with some one of which at least your President for the time being may be presumed to have a speaking acquaintance.

The front of the scientific army is long and straggling, and the lines of communication are thin and broken. If your President happens to have any news from the front he will probably be a welcome messenger among those whose duty keeps them elsewhere. If this is not too ambitious an aim, I propose on the present occasion to attempt a brief report on a recent successful movement by the Astronomical regiment which I trust may be of interest to their Mathematical comrades. I speak of it from the outset as a regimental movement; for though the chief credit is undoubtedly due to individuals, even the brief report I shall make will shew how much one owed to another, and he to a third: how different lines of advance converged and coalesced: and, in short, how the whole movement was *organised*, not in the artificial manner which man has come to associate with the word, but in the sense in which the advances of Nature, including those of Natural Science, are organised.

The mainspring of the movement was the old human desire to find some new thing, to make some discovery of a sensational nature. This impulse led Professor W. H. Pickering of Harvard to search for a new satellite to Saturn, and to search in a new way by the examination of photographic plates. Eight satellites of Saturn were already known, and had all been found by scrutiny of the heavens, with telescopes of gradually increasing power. No telescope was at Professor Pickering's disposal large enough to suggest any hope of success in finding a new satellite by this direct method; but the photographic plate enables us to use the time-integral of a telescope—$\int_0^T (\text{telescope})\,dt$ instead of (telescope)dt merely—and further, the upper limit of the integral is, in these days of dry plates, at our disposal. Hence a small

telescope acquires a new dimension, which may be so extensive that its quasi-volume is comparable with that of a large one. Profiting by this development Professor W. H. Pickering found a ninth satellite of Saturn in 1900, and named it Phoebe. But a terrible mortification was in store for him. He had followed the satellite on his photographic plates long enough to determine an approximate orbit and to predict its place when Saturn should next be favourably situated for observation, and unfortunately no satellite was there! The new plates were examined with the greatest care; the computations were checked again and again; the old plates were re-scrutinised; no clue to the mystery was forthcoming, and the little satellite seemed only to have been found to be lost again; indeed, some not very complimentary remarks were made about "bogus discoveries," by astronomers who had not seen the original plates.

But Professor W. H. Pickering is a man of considerable resolution. He knew the satellite was there, and he set himself to find it again, even on the plates which seemed to disprove its existence. He cast his net wide, and after a long and toilsome piece of dredging, at last he netted that of which he was in search, in a totally unexpected place—so far from his prediction that for some time he thought that it must be another satellite. Could any conceivable way be suggested for reconciling the apparently contradictory facts? Was there any possible ambiguity in the first solution?

Probably most of my hearers have spent tears (whether their own or their pupils') on the "ambiguous case" which occurs in the solution of triangles. They know from grim experience that a side of a triangle may present itself as a straddling pair of legs; making either an acute or an obtuse angle with the base. Almost any independent information about the triangle, such as the sequence of the angles in magnitude, would enable us to settle which leg to use; but without any such help we are left in complete doubt. Now similar ambiguous cases occur in the geometry of the planetary and stellar motions, and sometimes it is impossible to resolve the doubts. In the case of a double star for instance, we see one component revolve round the other in an ellipse, as it should do according to Newton's law of gravitation. The fact that the central star does not occupy the focus of the ellipse we ascribe unhesitatingly to the effects of projection, the plane of the orbit being inclined to the line of sight at some casual angle; and it is immediately suggested that we can determine this angle from the observed discrepancy between the real and apparent focus. But on attempting to do so, we find just such an "ambiguous case" as in the solution of triangles; the plane of the orbit appears as one of a pair—not perhaps of legs but of wings, let us say (for the equivalence of

wings and planes is becoming tolerably familiar): we cannot tell which of the pair it is. And this doubt carries with it an uncertainty as to the real direction of revolution of one star round the other. If the left wing is correct, then the movement is clockwise; if the right, anti-clockwise. Any independent information would enable us to decide. Thus the spectroscope will tell us whether at any given moment one component is approaching us or receding from us; and if we know this, one of the alternative suppositions is immediately struck out. But when the stars are faint it is not easy to make this spectroscopic observation.

A planet and its satellite are much like a pair of stars; but there are some important differences. In the particular case of Phœbe, the spectroscopic observation is quite impossible owing to the extreme faintness of the satellite; but fortunately there was another way of deciding between the two wings. It will be tolerably obvious that a pair of wings could only appear as a single wing from a particular direction; shift the point of view and we see the two wings separate, although we could of course rearrange them to appear single in the new direction of vision. Now the perspective of a double star never changes; the movements of the stars are so puny compared with their vast distances that for ages the stars were regarded as fixed in direction; and even to-day this is true for many purposes.

But it is quite otherwise with the planets; even a distant planet like Saturn changes its position in the heavens sensibly from year to year; and hence we get new views of its satellites which enable us to construct the orbits without ambiguity. Professor Pickering had chosen the wrong wing in computing the orbit of Phœbe; if we had not changed our point of view this would not have mattered, for either wing would serve indifferently; but from a new point of view the mistake was made manifest, firstly by the failure of the original predictions made from the wrong wing, and secondly, and more completely, by the satisfactory success of revised calculations made with the right wing.

But how came it that he was not alive to the ambiguity? If only he had taken account of it he might have saved himself much anxiety and labour; but he did not: he proceeded on the assumption that he could identify the proper wing without any question, and could scarcely persuade himself to try the alternative even when it occurred to him to do so; and there need be no delicacy about mentioning this apparently wilful blindness, because it may safely be said that any other astronomer would have adopted the same attitude; and so far from any special blame attaching to Professor Pickering, we must accord him the special credit of breaking down a well-established prejudice. He

assumed, in accordance with this prejudice, that all the satellites of Saturn were revolving in the same direction round the planet. This was true for the eight already known (and not only for them but for all the other satellites, with barely an exception, and for the movements of the planets round the Sun); what more natural than to suppose that it was true of the ninth? But it was not so; incredible as it at first seemed (I well remember the shocked surprise with which I first heard the suggestion)—incredible as it seemed, Phoebe, the outermost satellite, is revolving in the direction contrary to that of all the others. It has been already remarked that the two wings correspond to opposite directions of revolution; and hence it is clear why the very natural assumption misled Professor Pickering entirely.

To follow the consequences of this discovery would take us too far from the main thread which I wish to follow at this moment. I can only briefly mention the suggestion of Prof. Pickering that Saturn himself had originally rotated in the reverse direction and that his daughter Phoebe was born in this early period: that then the planet gradually turned over, so as to reverse his apparent direction of rotation; and in this new position shed the younger eight satellites. This suggestion was confirmed by the mathematical investigations of Mr. F. J. M. Stratton, who found a sufficient explanation of the turning over in tidal action. It is perhaps worthy of our attention in passing that Mr. Stratton found the mathematical procedure ready to his hand in the work of Sir George Darwin, who had investigated changes in the inclination of a planet's axis to the plane of its orbit, but had not pushed the investigation beyond the point at which it became zero. Apparently it had not occurred to him that a planet could turn over: the solar system had not at that time provided any serious suggestion of the kind, and he was content with the guidance of known facts. We may put this alongside the trustfulness of Prof. W. H. Pickering in the matter of the direction of revolution of Phoebe; and we may draw from both incidents the inference that our imagination is too apt to be fettered by the facts familiar to us. This inference has doubtless been drawn before, and will be drawn again.

But we must resume the main thread of the story.

The dramatic discovery of Phoebe, and the sensational reversal of two well-established assumptions as to the behaviour of planets and satellites, naturally brought other enquirers into the field. Were there other satellites like Phoebe, attending on Jupiter for instance? Provisional answers were forthcoming almost immediately. Within a few months of the rediscovery of Phoebe, Mr. Perrine, of the Lick Observatory, announced two new satellites of Jupiter in rapid succession. Four satellites had been found by Galileo with his original telescope in 1610; the

fifth, a tiny inner satellite, was discovered by Prof. Barnard with the great Lick telescope in 1892; the sixth and seventh were found at this same observatory from photographic plates following the lead of Prof. W. H. Pickering. It was realized that these new satellites resembled Phoebe in being at a much greater distance from their planet than those already known: further, there was the curious fact that their orbits were nearly the same; but whether one or both were moving in the reverse direction round the planet could not be settled at once. It was necessary to wait until the point of view had sensibly shifted, and when at last the question was settled, it was found that both were moving in the usual direction. This was a distinct disappointment, for there had seemed to be a double chance of finding an analogy to Phoebe, and yet both chances had failed.

We now come to the third chapter in the history. So far the attack had been conducted almost entirely in America, with the fine instruments and climates which have become famous. The only British share in the fighting had been this: that the telescope used by Prof. Perrine had originally belonged to Dr. Common of Ealing. But he did not make the mirror himself, as he later made his five-foot mirror: he merely mounted and used it, and then sold it to Mr. Crossley. Moreover, even if he had made it, so much of the telescope had been renewed (lock, stock, and barrel) that he could have claimed no share in the discoveries beyond this: that if he had never had the telescope, it would never have found its way to the Lick Observatory, and so the discoveries might not have been made. But we need not dwell on such speculations, for Dr. Common left a much more substantial claim to a share in this campaign in the beautiful 30-inch mirror he made for the Royal Observatory at Greenwich. When the new tiny satellites of Jupiter were found in America, the Greenwich observers were keen to know whether they could photograph them with this beautiful instrument, and to their great delight they succeeded. It was not only the instrument which was on its trial: our national climate was also concerned, and both have survived the ordeal. I well remember a visit paid by Prof. Perrine to Oxford on his way home from the eclipse of 1905, when he gave it as his opinion that though they might succeed in photographing the sixth satellite at Greenwich, they would not get the fainter seventh, owing to the climate. Within a few hours of the utterance the success was assured, and since then it has been part of the regular work at Greenwich to photograph both the sixth and seventh satellites, and to map out their orbits. We have learnt to think more respectfully of our English skies; and the lesson has, I hope, not been confined to ourselves.

But a more splendid vindication was to come. In the course of the regular work just mentioned, Mr. Davidson and Mr. Melotte not only took stock of these known satellites, but kept their sharp eyes open for any new one that might still be undetected; and on one fortunate day their vigilance was rewarded. To Mr. Melotte fell the coveted prize: on February 29, 1908, he found an eighth satellite, far outside the sixth and seventh, and in course of time it became clear that Phoebe was no longer unique, but that Jviii was also revolving round his primary in the unusual direction. We infer that Jupiter must, like Saturn, have been revolving originally in the reverse direction and have turned over; indeed, Mr. Stratton's work shewed that all the planets have probably turned over at least once, and some of them are perhaps in course of a second somersault. The old ideas that the planets and satellites *all* twist and turn in the same direction as the Sun, and have done so *from birth*, must be abandoned entirely.

By the kindness of the Astronomer-Royal I am enabled to shew you a beautiful model of the Jovian system as we now know it, and you will see what sensational additions to our knowledge have come within the last few years. In the middle are the four Galilean satellites and Barnard's fifth: far outside these are the interlinked orbits of Perrine's sixth and seventh, and again far outside these is the complicated orbit of Melotte's eighth. And this brings me to the last chapter of the story, which to a mathematician is the most interesting of all. The model shews how unlike is the path of Jviii to anything we have hitherto visualized as an orbit. We have been accustomed to think of an orbit as closely resembling an ellipse: we know that disturbances from bodies other than that which occupies the focus are continually modifying the ellipse, but we have hitherto thought of such changes as small—for many purposes negligible.

In the case of Jviii they are, however, so large that the curve is no longer even closed. After what should be a complete revolution the satellite does not return to the same spot at all; and if it ever does come back again it is only after a large number of interlaced turns. Different turns round Jupiter are beautifully picked out in the model by different colours, and the eye is thus aided in following the complex movements of this new nephew of ours.

The reason of the complexity is not far to seek. When a satellite is close to Jupiter, his movements are almost entirely directed by the planet. If we removed the same body far enough away from Jupiter, the attraction of the planet would become so feeble that the Sun would assume the direction, and the body would leave Jupiter and revolve round the Sun as a separate planet. At distances intermediate between these the Sun and

Jupiter struggle for mastery, which may under certain conditions be divided between them. These conditions do not actually apply to the new satellite, which is too close to Jupiter for the Sun ever to get the upper hand, but not too close for him to exert serious disturbance, so serious that we are brought face to face with the well-known "problem of three bodies" round an unexpected corner. The difficulties of this classical problem are as well known as its name, and we have now to see how they have been surmounted in this sudden and unforeseen emergency.

But for this we must leave the thread we have so far been following and take up another. Where the end of this other may really be I shall not attempt to say; but Sir George Darwin, who has spun a good piece of it, traces it back to Lord Kelvin, and we may be content to follow it from there forwards. In his *Popular Lectures* (vol. i., 2nd edn., pp. 31-42) Lord Kelvin suggested that a curve might be drawn from a knowledge of its curvature. He returned to the subject in the *Philosophical Magazine* in 1892 (vol. xxxiv., pp. 443-447), and gave one or two examples. His method is diagrammatic and quite elementary, amounting to very little more than what would occur to any mathematician directly the idea is mentioned, viz., starting with any point P_1, calculate the radius of curvature CP_1, and take any point C_1 at this distance from P_1: describe a short arc P_1P_2 round this centre: calculate a new radius P_2C_2 for the new point P_2 and repeat the procedure; *something* of this kind would occur to anyone: Lord Kelvin does indeed make a detailed suggestion that the curvature should be calculated, not for the *beginning* of each arc, but for about the middle, which can be estimated closely from the knowledge of the previous arc; but this again would probably occur to many people directly they set to work. The chief merit of Lord Kelvin's suggestion seems to be that which he shared with Columbus: none had previously thought of carrying it out. *He* first thought of carrying it into practice: and more than this, *he* first thought of applying it in the domain where it has been so conspicuously successful, viz. that of gravitational astronomy. For in his paper of 1892 he takes as one example the motion of our moon under circumstances different from those which actually hold, working out a curious triple figure-of-eight as a possible orbit.

Now, there, so far as I know, he left the matter; but Sir George Darwin took it up, and by his great memoir on "Periodic Orbits" in the *Acta Mathematica* (vol. xxi.), he made us familiar with the wider possibilities of planetary movement.

Up to that time orbits had always been treated as slowly changing ellipses,—or if not ellipses, some other curve which only slowly changed its elements. The opening sentences of Sir G. Darwin's memoir indicate that he only with great re-

luctance left this traditional procedure behind for the new one indicated by Lord Kelvin; but he saw no way of retaining it in the attack on certain problems he wished to attack, and was almost reluctantly driven to the new procedure, with its laborious numerical work. But his decision was justified by the results: new ground was broken, and our acquaintance with possible orbits much extended. The work seemed to have very little relation to actual facts as we knew them—only in one instance did it bear fruit of a tangible kind by giving a possible explanation of the Gegenschein—but this did not deter a man of the scientific faith of Sir George Darwin. He was content to wait for results, direct or indirect, and a most striking indirect result of his patient work concerns us to-day.

We must now return to Jupiter's eighth satellite, which was discovered at Greenwich on a plate taken on Feb. 28th, 1908, though once found it was recognized on previous plates for a month back. At first it was not known whether it was a satellite or a small planet, but it was kept under the most diligent observation, and with splendid promptitude Dr. Crommelin announced to the Royal Astronomical Society at its April meeting that the object was probably a satellite, revolving like Phoebe in the retrograde direction and at a great distance from Jupiter. But he realized already the great difficulties of calculating its orbit by the ordinary methods: the sun's attraction (as ultimately proved) might be as much as 10 per cent. of that of Jupiter; and in fact a situation had at last arisen in actual practice similar to those hypothetical cases to which Sir George Darwin had drawn attention. Dr. Cowell, with whom Dr. Crommelin was collaborating in work on Halley's comet, and to whose mathematical skill he appealed when this new situation arose, at once thought of Sir G. Darwin's procedure as possibly applicable. Now I hope it is doing Dr. Cowell no injustice to say that it is less easy for him to make himself acquainted with the exact details of a method than to devise one of his own from a general knowledge of what is wanted. He is not altogether singular in this respect, and after all we may reap (as was the actual case in this instance) the advantage of getting two methods instead of one. In the present case Dr. Cowell presented the idea of Lord Kelvin, not in Sir G. Darwin's dress at all, but in a beautifully simple and new one. I venture to reproduce it here so that you may actually have it before your eyes without the trouble of looking up anything at all, though his original paper, "Monthly Notices," R.A.S., June, 1908, is well worth reading and very short.

Let the co-ordinates (x, y, z) of a particle be supposed expressed in powers of the time (t); and at time $n+t$, let

$$x = x_n + a_1 t + a_2 t^2 + a_3 t^3 + \text{etc.}$$

Now, we shall suppose t varied by successive units, and it is easily proved that $x_{n+1} = 2x_n - x_{n-1} + 2a_2 + 2a_4 + 2a_6$.

When (x, y, z) is given we can calculate (X, Y, Z) the accelerations of the particle due to attracting masses in known positions. (Note that the movements of the gravitating masses which attract the particle are presumed to be known; *i.e.* the particle is of zero mass and does not influence them by its own attraction; hence this is *not* the complete "problem of three bodies," or rather n bodies.)

But then $$X = \ddot{x} = 2a_2 + 6a_3 t + 12a_4 t^2 + \ldots$$

therefore $$X_n = 2a_2$$

and $$P_n = \tfrac{1}{12}(X_{n+1} - 2X_n + X_{n-1}) = 2a_4 + 5a_6 + \ldots.$$

Thus $$X_{n+1} = 2x_n - x_{n-1} + X_n + P_n + (3a_6 + \ldots).$$

Now, if the unit interval of time is sufficiently small the remainder $(3a_6 + \ldots)$ can be neglected; it will not affect the last decimal place used. We can thus find x_{n+1} when we know x_n and x_{n-1} and X_n and P_n. The only difficulty is with $P_n = \frac{1}{2}(X_{n+1} - 2X_n + X_{n-1})$ for which we must know X_{n+1} and therefore $(x_{n+1}, y_{n+1}, z_{n+1})$. But a close value of X_{n+1} can be obtained from an estimated value of these co-ordinates; and the test of its being close enough is that when x_{n+1} is actually calculated it must agree sensibly with the prediction, and, if not, another approximation must be made.

This is practically all. It is seen at once how we can calculate the successive positions of the particle step by step, making each step secure before proceeding to the next. It reminds one of the way in which an Alpinist scales an ice wall. There is no elegant analysis, no dexterous manipulation of complicated formulae; the actual equations of motion

$$\ddot{x} = X,\ \ddot{y} = Y,\ \ddot{z} = Z$$

are used as they stand, without even the single integration of which they are capable. But there is no doubt of the effectiveness of the method. Quite apart from its success in the case of Jupiter's viii satellite, another signal proof of its value has been given in the case of Halley's Comet. It was remarked above that Drs. Cowell and Crommelin were already collaborating in work on the orbit of Halley's Comet at the time of the discovery of Jviii. Dr. Cowell had already devised some new mathematical methods for shortening the labour of following the Comet in its long journey of 75 years between apparitions. But the effectiveness of the procedure devised for Jviii was so convincing that they determined to apply the same method to Halley's Comet in order to predict the circumstances of its return in 1910 as accurately as possible. The result was conspicuously satisfactory; the Comet was detected by Dr. Max Wolf, of

Heidelberg, early in September, 1909, on one of his photographic plates within a few seconds of arc of the predicted place, and then so faint that without a knowledge of its place it could not have been identified. This was sufficiently demonstrated by the fact that it was subsequently found to have been photographed already on August 20th by Mr. Knox-Shaw in Egypt, using another mirror made by Dr. Common, and presented with a mounting to the Egyptian Government by Mr. J. H. Reynolds, though the speck on the plate could not be identified as the Comet without independent information.

A few details may bring home to us more clearly the kind of work involved in this prediction of the Comet's place. The whole previous observation of the Comet was first condensed into a very powerful essence. The investigators were content to extract the place of the Comet (three co-ordinates given to eleven significant figures each) on two dates in 1835. Step by step they then found its position at consecutive intervals of two days until it was far enough from the Sun to allow them to use four-day intervals without sensible error; these were continued until they could be changed to eight days, then to sixteen, and so on to 256 days, which was the longest interval used. To cover the whole journey of 75 years to 1910 some 300 steps were necessary. The most laborious part of the work was of course for the path near the Sun, where eleven significant figures were used. Let us consider what this means. The co-ordinates represent distances comparable with that of the Earth from the Sun, say 100 million miles, which has 8 zeros. Eleven places would thus take us to the thousandth part of a mile, say 5 feet. The calculations were therefore of such refinement as to take account of the difference in the Sun's attraction on the opposite sides of an ordinary table.

We have followed the genesis of this new method historically in the past; it would be rash to speculate on its future. The necessary labour is a great drawback: on the other hand, the simplicity and effectiveness are surely a sufficient counteraction. Anyone in this room, who can add up and use logarithm tables, could start at once with confidence on the solution of any suggested case of cometary movement among a prescribed system of planets, and would have a good chance of finding some curious and interesting things of a quite new kind. For instance, if a particle were started from midway between Jupiter and Saturn, what would happen to it under different conditions of projection? Or take Jules Verne's problem of a projectile from the Earth towards the Moon. We know that his own solution is wrong, but what is the right one? No one has hitherto worked out such problems, because the analytical method is so difficult. We have learnt the lesson that there is a simple and obvious way to go to work, the only disadvantage of which is the labour required.

Perhaps some system of friendly co-operation might make even the labour itself an amusing game rather than a toil. Among a company of mathematicians there may be some to whom such a suggestion is welcome.

I have related the facts leading up to this new though simple departure without much in the way of comment; but in reviewing them a reflection or two may be pardoned. First, I venture to hope that I have made good my position that the gratifying result is the outcome of an organized movement rather than the achievement of an individual. To make such a statement is to run some risk: we remember that Airy said something of the same kind about the discovery of Neptune, and did not improve his own position by the statement. But it is inevitable that with the march of time individual prowess should yield to combined action. Should Ajax or Achilles appear again they would not find the same opportunities for the glorious deeds which Homer sang: the glory is for the man who best combines his forces and utilizes opportunities as they arise, and even when he actually leads his men into battle his own sword may count for so little that he is more effective (as Gordon at any rate believed) with a cane. Something of the same kind is true in Science; at any rate in the incidents we have reviewed we have seen how the discovery of a satellite by one investigator stimulated others to search for more; how the following of these by yet other workers again led to a fresh discovery of unusual interest; how the suggestion of Lord Kelvin that his students should draw curves from their curvature led Sir George Darwin to the investigation of Periodic Orbits, and thus to the use of the method by Dr. Cowell for the new satellite of Jupiter; how this again led to the extended application of the method to a comet, and probably to its general use in the future.

We have found it necessary to mention many names: W. H. Pickering, Stratton, Perrine, Common, Davidson, Melotte, Crommelin, Cowell, Max Wolf, Reynolds, Knox-Shaw, G. H. Darwin, Kelvin; I do not say for a moment that merit attaches to all of these in equal portions, but merely that merit attaches to all. I do not say for a moment that there is not a scientific Ajax or an Achilles among them, but only that many of them were dependent on opportunities created by their colleagues. It may perhaps be urged that this does not apply to Prof. W. H. Pickering or to Lord Kelvin, both of whom broke fresh ground; but even they were dependent on their successors for the reaping of the full harvest.

A second remark, which I also make with some fear of misunderstanding, is that in entering on any new line of research we often lose rather than gain from what is already known. It would have saved Prof. W. H. Pickering much mortification and

labour and anxiety if he had *not* known that satellites usually revolve in the same direction round their primaries: trusting to this knowledge he was led quite astray. It seems possible also that the fact of the planets all rotating on their axes in this direction (with negligible exceptions) caused Sir George Darwin to close an interesting research at a point when it would have been profitable to continue it. But the most striking instance of all is the use of the ellipse as a starting-point for planetary motions. Newton's great discovery that this is the orbit for a pair of bodies has led to its retention in the calculations in many cases where it is better away; and it has required the combined courage of three or four original thinkers to break away from the shackles of our previous knowledge.

Thirdly, it may save misconception to remark that the new departure is one simply of practical procedure, and does not represent any new mathematical method. A genuine mathematical method would give us the place of an object at any epoch however distant without the necessity of passing through intermediate epochs. It is true that the error of the prediction would probably be large if the epoch were in the far future; but this defect is common to all methods whatsoever, since it originates with the imperfection of the data. Starting from necessarily imperfect observations made over a limited period, we could not by any method foretell the future without gradually accumulating error. But putting this necessary disqualification aside, a mathematical procedure would foretell the whole future, and summarize the whole past, in an analytical expression. The method of quadratures employed with such success by Dr. Cowell and Dr. Crommelin makes no claim whatever of this kind: it can give us no information at all about a future situation until all the intermediate history has been accurately compiled. One possible exception may be mentioned only to be dismissed: it is conceivable, though unlikely, that all the bodies concerned in the calculation should simultaneously return at some time to positions they formerly occupied, and then, of course, history would repeat itself indefinitely. But the return must be so exact as to be incredible: the slightest deviation in any one of the bodies would create differences which there is *as yet* no means of calculating except that of going through the whole work again. It is not impossible that some simpler alternative may be devised: indeed our mathematical training disposes us to believe that there must be some way of deducing from the complete calculation for one set of data the corresponding results for a slightly different set; but so far as I know, this matter has not yet received attention. However, the establishment of the arithmetical procedure as a valuable practical method will doubtless lead to the study of the problems connected with it.

Fourthly and finally, I venture a word of satisfaction at a new proof of the great value of rectangular co-ordinates. The simplicity and beauty of the method depend largely on the use of the three equations

$$\ddot{x} = X, \ddot{y} = Y, \ddot{z} = Z$$

instead of those into which they can be transformed where the radius vector and its angular co-ordinates are taken as variables. In my opinion Astronomy has suffered much from the incubus of spherical trigonometry as crystallized in our text-books: the formulae are so distressingly ugly. The use of photographic plates introduced us to opportunities of substituting formulae in rectangular co-ordinates for those of spherical trigonometry; for an astronomical photograph represents the projection of the heavens on a plane instead of on a sphere. In consequence it became possible to use much simpler formulae than before, although it was at first difficult to persuade astronomers, who are a conservative race, of the advantages within their grasp. Even in the case of photographs of the Moon, which is actually a curved surface, Mr. Saunder has found it a great simplification to use rectangular co-ordinates in his computations, and has thereby completed for the first time an adequate fundamental survey of the Moon, whose surface we may now be said to know better than that of the Earth itself.

It does not seem to me hopeless that the work of measuring astronomical photographs might ultimately find its way into the educational curriculum as an easy introduction to work of the greatest possible refinement. It would render the introduction easier if some preliminary elementary knowledge of the chief facts of Astronomy could be introduced first. The old "use of the globes" avoided the ugly formulae of spherical trigonometry, but has dropped out of use, partly because it was not considered accurate enough to merit the attention of mathematicians, and partly because, after all, work with a globe has some inconveniences. Recently we have lost some of our contempt for work of an approximate character: we use four-figure logarithms instead of seven-figure; and do much work on cross-ruled paper. Perhaps work with the globes might again find its way into the classroom. And I may perhaps draw attention to an alternative which removes some of the inconveniences of work, viz. the use of a cube instead of a sphere. The faces of the cube are flat and can be opened out into a single plane if required. All great circles become broken straight lines, and a simple rule enables us to continue a straight line of one face into the next. Moreover (as is shewn in "Mon. Not," LXX. p. 204) *metrical* properties can be preserved by the use of a very simple divided scale: that is to say, the sides and angles of any spherical triangle can be measured, and the exercise should form an instructive extension

of the simpler work on plane triangles. If more advanced work still is required, the measurement of astronomical photographs with a simple form of microscope-micrometer is capable of furnishing work not only of interest to the student, but of permanent value to the astronomer.

But I fear I have betrayed the desire to grind a very obvious axe, and I had better cease before your politeness is exhausted.

Mr. Goodwill read a paper on the Teaching of Mechanics, after which the Meeting was adjourned for lunch.

AFTERNOON MEETING.

THE REPORT OF THE COUNCIL FOR 1910.

The meeting was resumed at 2 o'clock and the President at once called upon Mr. Pendlebury to read the Council's Report for the year 1910. The report was as follows:

It is with much regret that the Council record the loss to the Association by death during the past year of three members of long standing—Mr. H. W. Eve, Dean of the College of Preceptors and formerly Head Master of University College School, an original member who joined the Association just forty years ago; the Rev. Robert Harley, F.R.S., an eminent teacher of former times, who had been a member of the Association for thirty-three years; and Mr. R. W. Hogg, of Christ's Hospital, at one time Fellow of St. John's College, Cambridge.

The Roll of Members of the Association continues to increase both in length and in influence; 94 members have been elected since the last annual meeting. The Association has now 620 members, of whom 9 are honorary members, 30, life members by composition, 55, life members under the old rule which has been rescinded, and 526, ordinary members. In addition it has about 190 associates attached to one or another of the branches. The list of members and associates now therefore contains more than 800 names.

During the past year the Council have received with approval a suggestion that a series of biographies of eminent mathematicians should be prepared for publication in the *Mathematical Gazette*, and that the best portraits that can be obtained of the subjects of the biographies should be reproduced in the *Gazette* in an attractive form. The matter is now in the hands of the Editor.

The Council have had under consideration also the manner in which the Association can help best to further the objects of the International Congress of Mathematicians to be held at Cambridge in August, 1912. The following suggestions have been made: (1) that an exhibit of models, books, and perhaps schoolwork,

illustrative of mathematical education in the United Kingdom, be arranged; (2) that a special number of the *Mathematical Gazette* containing an account of the progress of mathematical education in recent years, be prepared for presentation to visitors at the Congress; (3) that with the special number of the *Gazette* be issued a reproduction of some small and interesting mathematical manuscript or pamphlet, if any suitable work of the kind can be obtained; (4) that a collection of portraits of eminent British Mathematicians be made; and (5) that publishers be invited to arrange for an exhibition of typical mathematical books. The Council would welcome further suggestions with the same object. In the meantime a small committee of the Council has been formed for the purpose of conferring with the Cambridge Committee, which is responsible for the management of the Congress.

The Committee of the Association, formed to consider the teaching of elementary mathematics, have recently drawn up a draft report on "The Teaching of Elementary Algebra and Trigonometry." A copy of the draft has been sent to every member of the Association and to every associate, and the matter will be brought up for discussion at the Annual Meeting. The Report will be reconsidered by the Committee in the light of this discussion.

The local Branches have made good progress during the year.

In a communication received from Professor H. S. Carslaw of the University of Sydney, he states that at a meeting held at Sydney on the 27th of October, 1910, it was decided to form there a branch of the Mathematical Association, and that 30 members were enrolled at the preliminary meeting. Professor Carslaw will act as chairman of this branch.

The meetings of the North Wales Branch have been continued during the year, and have been well attended. Various questions relating to the teaching of mathematics in schools have been profitably discussed.

The Southampton and District Mathematical Society now consists of 25 members, drawn from Southampton, Bournemouth, Winchester, and the Isle of Wight. Four meetings have been held during the year; the papers read have been much appreciated, and have given rise to keen discussion. In consequence of the success of the year's work it has been decided that in future the meetings of the Society shall be held once a month. Among the subjects discussed during the year have been Ancient and Modern Methods in Geometry, Axioms and Postulates in Geometry, Logarithms, and the Board of Education Circular.

The London Branch was formally inaugurated in February, 1910, and has grown very rapidly. It now includes 130 full

members of the Association, and 123 associates. Four meetings have been held, at which the attendance has averaged about 130. Among the subjects discussed have been the Board of Education Syllabus of Geometry, Graphs, Elementary Arithmetic, Irrational Quantities, and Specific Volume. A social gathering, which was held at the Mercers' School on the 10th of December, was well attended, and was much enjoyed.

Professor H. H. Turner, F.R.S., retires at the Annual Meeting from the office of President, which he has held for the last two years. His period of office has been one of great activity in the Association, and will be memorable in its history. The progress which the Association has made since 1908 is largely due to his vigorous support. The Council have much pleasure in nominating Professor E. W. Hobson, Sc.D., F.R.S., to be President of the Association for the years 1911 and 1912.

Dr. F. S. Macaulay and Mr. D. B. Mair now vacate their seats on the Council in compliance with the rules of the Association, and the members present at the Annual Meeting will be asked to nominate and elect two other members to succeed them.

The Council again desire to express their hearty appreciation of the manner in which Mr. Greenstreet has continued his laborious work as Editor of the *Mathematical Gazette.*

The Council desire also to express to the authorities of King's College and of the London Day Training College, their warmest thanks for the kindness and hospitality which they have received at their hands, and for the accommodation which has been afforded them for the meetings of the Council, of the Committees, and of the Association itself.

The report was adopted, and the election of Officers and Council for the year 1911 was then proceeded with. Professor E. W. Hobson, Sc.D., F.R.S., was elected President; Mr. F. W. Hill was re-elected Honorary Treasurer, and Mr. C. Pendlebury, Mr. H. D. Ellis, and Miss E. R. Gwatkin were re-elected Honorary Secretaries; Mr. C. V. Durell (Winchester) and Mr. P. J. Harding were elected to the vacant seats on the Council.

The "business" meeting terminated with a vote of thanks to the retiring president, proposed by Mr. Siddons who mentioned that Professor Turner had shown his interest in the Association in the most practical way by attending every meeting of Council and Committee during his term of office. The proposal was seconded by Mr. Jackson, who said that it might be a doubtful compliment to describe Professor Turner as having always been ready to bring the Association news from the front—unless it were added that he contrived to do so without ever leaving the firing line. Professor Turner had secured the rapid dispatch of business, and yet managed to convey to every member of

Council or Committee the conviction that their arguments had been patiently and sympathetically weighed.

The vote was carried with acclamation and acknowledged by Professor Turner. Professor Turner was then elected a Vice-President and Honorary Member of the Association.

THE MATHEMATICAL ASSOCIATION.

VERBATIM REPORT OF THE AFTERNOON ADDRESSES AND DISCUSSIONS AT THE ANNUAL GENERAL MEETING.

Held at the London Day Training College, Southampton Row, W.C., 11th January, 1911, the President, PROFESSOR H. H. TURNER, F.R.S., in the Chair.

AFTER the transaction of official business,

The Chairman said: We now come to the reading of papers, and I am glad to be able to call upon Canon Wilson to read his paper first. Canon Wilson is an original member of this Association; indeed, I am not sure that I should not be right in saying that he is *the* original member. I believe his book on new methods of teaching geometry gave rise to this Association, and it is a great pleasure to welcome him here, not only as an old member, but also as my old head master, who was very kind to me at Clifton. It is interesting to notice that he is still so vigorous as to be able to unearth manuscripts from the Worcester Cathedral Library, and I hope the Association will give him a hearty welcome.

ON TWO FRAGMENTS OF GEOMETRICAL TREATISES FOUND IN WORCESTER CATHEDRAL LIBRARY.

BY THE REV. CANON J. M. WILSON, D.D.

I believe I am the only surviving original member of that A.I.G.T., to which the President has referred (The Association for the Improvement of Geometrical Teaching). This Association arose out of a book which I wrote at Dr. Temple's (the late Archbishop of Canterbury's) suggestion, after the Commission of Inquiry into Secondary Education (the Endowed Schools Commission), that some new method, simpler than Euclid, should be brought into our geometrical teaching. It was very desirable that there should not be competition between private individual writers of text-books competing for public approval and public use, and so we formed the Association in order to unify the teaching on the new lines.

The paper that I am going to speak on to-day is one that I wrote for the Archaeological Society of Worcestershire, and is the result of an investigation into the bindings of books in the old Cathedral Library of Worcester. There is there an ancient monastic library of manuscripts ranging from the eleventh century to the fifteenth—about 270 volumes of actual working books. And pasted into the bindings of these books, or used sometimes in binding the registers of later books, were fragments of older manuscripts. I have just presented our new President with a photograph of the two manuscripts about which I am going to speak. One of them has been proved to be part of a treatise on geometry written about A.D. 980 by Gerbert or Gervase. The interest of this work is that Euclid was not known in any of the monasteries or cathedral schools at the time of its production, but only in Spain.

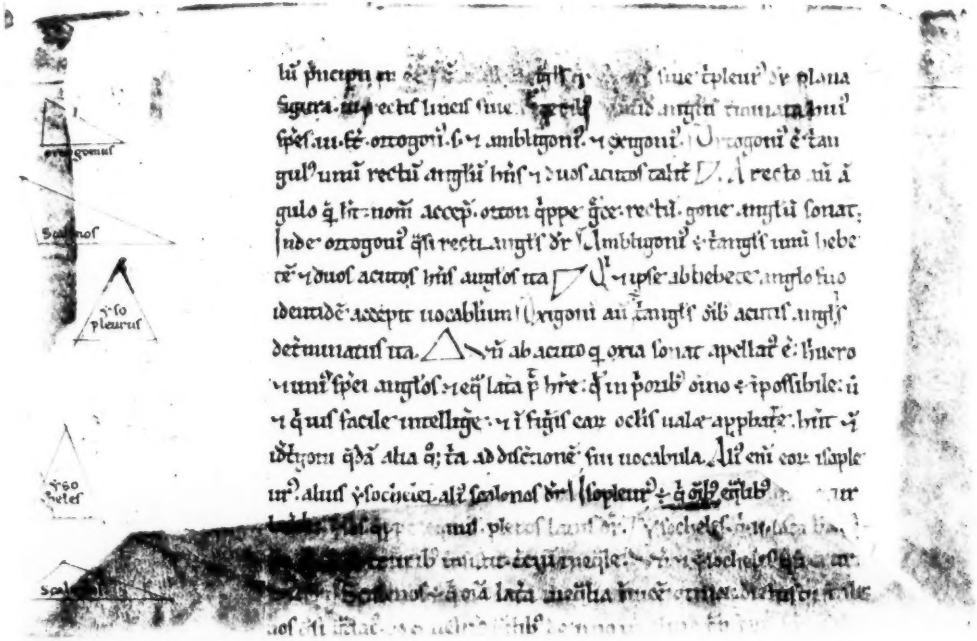
ON TWO FRAGMENTS OF GEOMETRICAL TREATISES FOUND IN WORCESTER CATHEDRAL LIBRARY.

As Librarian of the Worcester Cathedral Library I have examined the bindings of most of the books, to see whether fragments of other and possibly older works, in MS. or print, had been written on fly leaves or pasted on the boards. I gratefully acknowledge that the laborious search among the printed books was mainly carried out by the Misses Webb, of Greenfield, Worcester, and produced many interesting fragments.

Among them is a portion of a leaf [Frag. xi.] of a treatise on geometry, dating probably from the first quarter of the 13th century: and as such fragments are rare I have thought it would interest our Society to see a facsimile, and to have a transcription of the fragment, and to hear a little about it. It is beautifully written on parchment, in ink of wonderful quality. The size of the paper is $6'' \times 4''$. It will interest palæographers from some peculiar abbreviations. In the transcription I have to acknowledge some valuable assistance I have received from Mr. Cosmo Gordon, of King's College, Cambridge, and from Dr. Warner, of the British Museum.

I now give the transcription in full: ii recti hoc modo Si due recte linee sese invicem per alteram ducte secent, aut iiii rectos efficiunt angulos hoc modo +, aut ii exteriores totidemque interiores ex adverso sibi invicem equos reddunt, qui tamen quatuor rectis angulis sunt equales, hoc modo . Due recte linee equali a se invicem spatio inductione sua distantes, et in infinitum ducte nunquam invicem concurrentes paralelli scilicet eque distantes dicuntur, ita = Quod si recta linea ab una ad aliam ducta fuerit, aut rectos angulos iiii ubi tangit eas efficiet , aut totidem rectis equos, binos scilicet exteriores, binosque interiores sibi ex opposito invicem equales taliter Possunt quidem et alia nonnulla de lineis et angulis inveniri et dici; sed ingredientibus hæc sufficere putavi.

In omnibus ergo ut dictum est planis figuris quæ quidem angulate sunt unam vel duas vel certe omnes has angulorum species necessario invenimus. Unam ut omnes rectos angulos habeant, aut hebetes omnes, vel omnes acutos; duas vero alios rectos angulos habeant, alios acutos; aut alios hebetes alios acutos; aut alios rectos alios hebetes; omnes ut rectus et hebes et acutus, quod tamen rarius evenit ut in una aliqua inveniatur figura. Quod totum posterius in earum formationibus satis clarebit. Nunc jam de



Facsimile of fragment of early 13th century MS. of Geometry by Pope Silvester II. (Gerbert). Worcester Cathedral Library.

triangulo, qui in planis figuris natura*liter** *primus occurrit sequens ratio quæ videbuntur aggredi tentabit, quare satis cuipiam potest clarescere omnium planarum figurarum triangu*lum† principium esse. Est autem triangulus, qui et trigonus sive tripleurus dicitur, plana figura iii rectis lineis sive lateribus et totidem angulis terminata. Hujus species iii sunt, ortogonus scilicet et ambligonius et oxigonius. Ortogonius est triangulus unum rectum angulum habens et duos acutos taliter . A recto autem angulos quem habet nomen accepit; orton quippe grece rectum, gone angulus, sonat. Inde ortogonius quasi rectiangulus dicitur. Ambligonius est triangulus unum hebetem et duos acutos habens angulos, ita ; Qui et ipse ab hebete angulo suo identidem accepit vocabulum. Oxigonius autem triangulus omnibus acutis angulis determinatus, ita, △; unde ab acuto quod oxia sonat apellatus est. Hic vero et unius speciei angulos et equa latera potest habere, quod in prioribus omnino est impossibile, ut et quivis facile intelligere, et in figuris earum oculis valet approbare. Habent etiam idem trigoni quædam alia quoque tria ad discretionem sui vocabula. Alius enim eorum isopleurus, alius ysocheles, alius scalenos dicitur. Isopleurus est qui omnibus equalibus continetur lateribus; ysos quippe equus pleros latus dicitur. Ysocheles qui ii latera habet equalia qui in ‡ quasi cruribus insistit, tertium inequale. Unde et isocheles quasi crurium § dicitur. Scalenos est qui omnia latera inequalia invicem continet, dictusque scalenos quasi gradatus eo quod velud gradibus de uno in aliud transfertur latus.

It will be seen at once that this is not Euclid. It consists of notes and observations on classified geometrical facts. It is not a scientific geometry, arranged into theorems and problems, each logically demonstrated, and ultimately depending on definitions, postulates and axioms explicitly stated.

The fragment was identified for me by Dr. Warner of the British Museum as from the geometry of Gerbert or Gervase,—a name once famous, now almost forgotten.

Gerbert was a brilliant pupil in the Benedictine Abbey School of Aurillac in the latter half of the 10th century. He thirsted for knowledge; he went to Rome, and then to Rheims, at that time the most famous school in France. Then he went to Spain, in hopes of sharing in the learning of the Saracens, who however jealously guarded their universities against Christians. He became in succession Abbot of Bobbio, Archbishop of Rheims, Archbishop of Ravenna, and finally in A.D. 979 Pope under the name of Silvester II. He is not the Silvester of our calendar on December 31, who was a much earlier bishop of Rome.

The fragment before you is from Gerbert's *Treatise on Geometry*, written about A.D. 980, the extant portions of which have been edited from various MSS. by Bubnov as *Gervasii* (*Silvester ii*) *opera mathematica*. Berlin, 1899. I examined this book in the British Museum Reading Room. He acknowledges his obligations to Boethius, of whom I shall speak presently; he gives the definition of the word geometry; traces the origin of the science to Egypt; and gives reasons for studying it; viz., its charm, and educating power, and its leading to the contemplation of the works of God; and he quotes Augustine, who was himself quoting Plato, as saying that "the eye

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* The words in italics have been cut off from the bottom of the leaf, and have been supplied from other MSS.

† The facsimile opposite shews what follows. Figures shewing the different sorts of triangles are drawn in the margin. (Plate 1.)

‡ For *qui in* another MS. reads *quibus etiam*.

§ For *crurium* another MS. reads *equicrurium*.

of the soul which may be dull to earthly things may by these studies be raised to spiritual things."

He treats in succession of the point, line, angle, surface and solid. He gives a proof that the 3 angles of a triangle are together equal to two right angles. He gives a neat method for testing whether a right angle has been correctly drawn; and he solved the problem, to which I invite your attention, how to find the sides of a right-angled triangle, when its hypothenuse and area are given.

Before he became Pope he was the most famous teacher in Europe, a reviver of learning in many branches, *studiorum reparator*; his lecture room was crowded; *discipulorum turmæ artibus instruendæ* flocked to him. Music was his special forte. He is referred to as "*Gerbertus, cognomento musicus, qui et Silvester.*" But he was a brilliant teacher of all the subjects of the *quadrivium*, the advanced studies of the schools and universities. The *trivium* or elementary studies were grammar or rhetoric, dialectic and logic; the *quadrivium* consisted of arithmetic, music, geometry and astronomy. They are all named in the well-known verse *Lingua, tropus, ratio, numerus, tonus, angulus, astra.* Notes from his lectures were widely used; there are portions of his work in MSS. in Paris, London, Oxford, Cambridge and St. Petersburg, from which his works have been edited. We may now add Worcester to the above list.

Besides writing geometrical and arithmetical works, he invented and made terrestrial and celestial globes, and exchanged them for old MSS. of classical Latin writers, some of which are said to have been saved in this way from destruction. He also made the famous clock of Magdeburg, which was long preserved. One of his inventions was an organ, described as follows. *Mirum in modum per aquæ calefactæ violentiam ventus emergens implet concavitatem barbiti et per multiforatiles tractus æreæ fistulæ modulatos clamores emittunt.* We should more briefly describe it as a steam-organ. This he made at Rheims, where it survived 200 years.

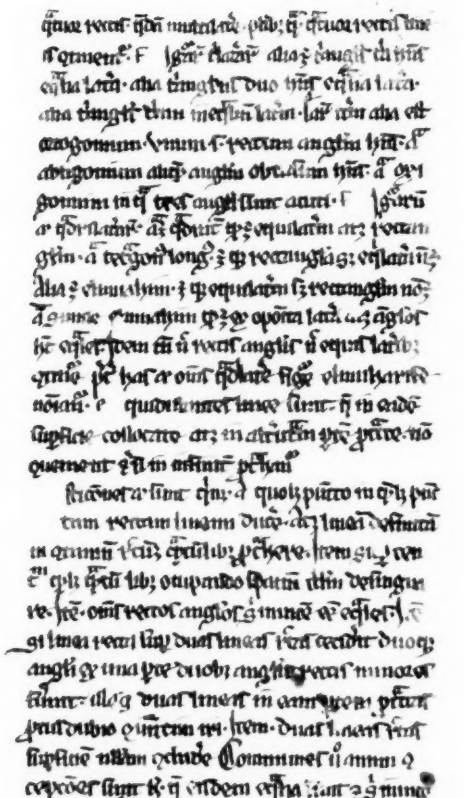
The world was divided as to the source of his amazing and varied knowledge. His friends said he was *divinitus directus*; but he was more widely believed to have sold himself to the devil. Enough has been said to shew that he was a very illustrious man upon whose traces we have come unexpectedly in our library.

When dying he composed and uttered, as his last words, a Latin verse, to say that his mathematics would not now help him. *Nil abacus mathesisque tibi, Gerberte, juvabunt.* It will come, I fear, as a shock to some of us to learn that a great Pope and mathematician thus passed away with a bad false quantity on his lips.

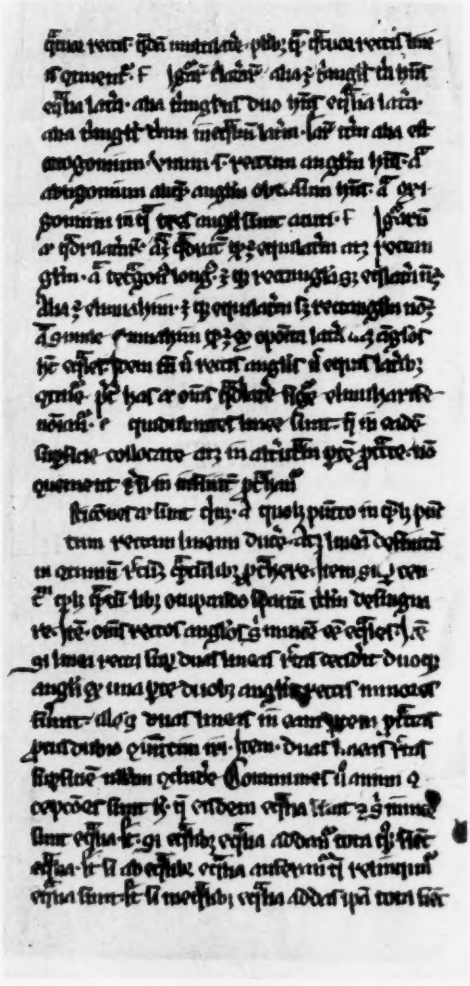
The other fragment (Plate 2) is written on two and a half blank leaves at the beginning of a volume of *Sermones et collationes* of the 14th century (Q. 89). The writing on each page occupies $4\frac{5}{8}'' \times 2\frac{1}{8}''$. It is at once seen to be a translation of Euclid. It begins with the familiar definitions of a point and line. *Punctus est cui pars non est. Linea est longitudo sine latitudine.* Later on among the definitions is a passage of special interest which I quote at length.* *Figurarum | autem quadrilaterarum alia est quadratum quod est equilaterum atque rectan | gulum: alia tetragonus longus est que rectangula sed equilatera non est: | alia est elmuahim: est quod equilaterum sed rectangulum non est: | alia simile elmuahim quia est* [*ex*] *opposita latera atque angulos | habet equales. Idem tamen nec rectis angulis nec equis lateribus | continetur. Præter hasce omnes quadrilatere figure elmuharife | nominantur.* I shall return to this presently.

On the same page (line 17) are the postulates. *Peticiones autem sunt quinque*: the three of Euclid as we know them, that we may join points,

* The passage quoted is from lines 7-14 of the facsimile opposite.



Facsimile of 2nd page of a Manuscript of Euclid, translated from Arabic, 14th century.
Worcester Cathedral Library. (2·89.)



Facsimile of 2nd page of a Manuscript of Euclid, translated from Arabic, 14th century.
Worcester Cathedral Library. (2-89.)

produce straight lines, and describe circles (the first two appear to be regarded as one postulate); and two more which in our Euclid we find among the axioms; viz. that all right angles are equal: and (line 22) *si linea recta super duas rectas lineas ceciderit duoque | anguli ex una parte duobus angulis rectis minores | fuerint illas duas lineas in illam partem protractas | procul dubio conjunctum iri.* Also that two straight lines cannot enclose a space. In other words the theory of parallels is based on a postulate, not on an axiom.

In the fifth line from the bottom of the page we have the axioms—*communes vero animi conceptiones scilicet quæ eisdem equalia sunt, et sibi invicem equalia sunt, etc.* On later pages follow the propositions. No figures are anywhere given; but there was a wide margin left which may have been intended for them. The familiar proposition is *Super datam rectam lineam triangulum equilaterum collocare.* The construction is as follows. *A duobus terminis date linee ipsam lineam ocupando cum circino duos circulos sese invicem secantes describe. Ab ipsa circuli sectione ad duos terminos linee proposite duas lineas rectas dirige.* And the proof is very brief. *Denique ex circuli descriptione argumentum elice.*

The fragment shews of course that Euclid was studied here in the 14th century. But the definitions quoted above illustrate a very interesting point in the history of geometrical teaching. We have in our Euclids the definitions of the square, and rectangle, and rhombus or rhomboid, and trapezium. But in our Worcester copy, these latter names do not occur; and instead of them we have the strange forms elmuahim, elmuharife. These are Arabic words; and it is a curious story how they got into a Latin translation of Euclid, who wrote in Greek.

The history is this. Boethius, whom I mentioned before, is often spoken of as the last Roman of note who studied the language and literature of Greece. He was a man of splendid character, and was murdered by his own officials whose rapacities he tried to restrain. This was in A.D. 526. Till the 13th century, when Aristotle supplanted him, the works of Boethius were the great educators of Europe. Among his works was a geometry founded on Euclid, but imperfect. It was known, as we saw, in the 10th century, that the mathematics of the Arabs were far in advance of those of Christendom; but they rigidly excluded Christian students from their universities. It was an Englishman, Adelhard or Æthelhard by name, a monk of Bath early in the 12th century, who passed the barriers. In his enthusiasm for mathematics he learnt Arabic, disguised himself as a Mahometan, and succeeded in entering in succession the universities of Cordova, Granada, and Seville. There he studied Euclid in Arabic, and translated it into Latin. But he had no words for rhombus and trapezium, and kept the Arabic words. His work on Geometry, of which this fragment is a portion, was written in 1120. He was a great traveller, visiting North Africa, Sicily, Greece and Asia Minor. One of his works is entitled *Perdifficiles quæstiones naturales.*

Such is the explanation of these Arabic words. The Greek text was never before him. He simply transcribed the Arabic words when he knew no Latin equivalent. His work, taken from the Arabic version of Euclid, was the foundation of all the editions of Euclid till 1533 when the Greek text was recovered.

The Arabic words appear variously spelt as might be expected. Elmuain, elinuam, elmuhahim, elmuhaim, helmunharife, elmdearifa, elmurifa are forms in MSS. in Vienna and Erfurt, differing slightly from our Worcester readings. There is a note on them in Bubnov's work above referred to on p. 176. Dr. Driver kindly tells me that Freytag gives el-mun-harif as the Arabic for trapezium.

It is a further and very interesting question whether this treatment of axioms and postulates is of Greek or Arabic origin. It has been suggested

that the Arabs may have had an earlier and purer text before them. The question is therefore analogous to that of the Septuagint and Masoretic texts of the Old Testament.

I hope that a sight of these rare fragments of the text-books of our Benedictine predecessors of the 13th and 14th centuries, and the glimpses they give of mediæval education, have been of interest to our Society, and illustrate some of the romance connected with even so prosaic a study as elementary geometry.

After reading portions of the above paper, Canon Wilson pointed out that Gerbert's treatise of the tenth century was a mere summary of geometrical facts, and that the systematic arrangement of those facts by Euclid was unknown to him. It is said that he endeavoured to visit the Moorish universities, but was refused admittance.

The second MSS., written in the fourteenth century, but copied from a work of the twelfth century is of peculiar interest both from its contents, and from the way in which it was obtained. An English monk of Bath, Adelhard or Æthelhard by name, early in the twelfth century, was so keen for mathematics that, as is said, he learned Arabic, professed to be a Mahometan, and passed through the Moorish universities, and translated into Latin the Arabic version of Euclid which he then studied. Some of the definitions, the postulates and axioms will be seen in the facsimile given in the *Gazette*.

Canon Wilson called special attention to two points. First, that in the definitions, where in Euclid we meet the words *rhombus* and *trapezium*, here in the MS. we find *Elmuhaim* and *Elmuharife*, Arabic words, for which Adelhard had no equivalents in Latin or Greek. Secondly, that there are in the MS. five postulates (*Peticiones autem sunt quinque*). The three of Euclid as we know them, regarded as two, one concerning making and producing straight lines; and the other concerning circles: the remaining three being what in our English Euclids are found among the axioms, viz. that all right angles are equal (*omnes rectos angulos sibi invicem esse equales*); the well-known axiom about parallels; and that two straight lines cannot enclose a space (*duas lineas rectas superficiem nullam concludere*).

Now, I do not know, Canon Wilson continued, what the philosophical mathematician of to-day will say about this. Which is right? Is it an axiom or a postulate that all right angles are equal? I am much too old to know. (Laughter.) What does the President think? The axioms, I may remark, are seen in the last five lines of the facsimile, and are called *communes animi concepciones*.

Mr. Bushell: Canon Wilson is not right in supposing that he is the only original member of the Association present, for happily I also can lay claim to that distinction: and, in response to the

Chairman's invitation, may I be allowed to convey to Canon Wilson our gratitude for his having come to us to-day? This is, I think, a very interesting epoch in the history of this Association, for, if I am not mistaken, next Tuesday will be the fortieth anniversary of that memorable first meeting which we held at Rugby, with Canon Wilson in the chair. He has appealed to you, Sir, to explain on the spur of the moment certain very difficult and much controverted points in elementary geometry. I remember very well that when he was in the chair in 1871 we heckled him in the same way. I have here in my hands the agenda of our proceedings on that day. We assembled, I may say, to bury Euclid. We did not think he would have a life of more than six months at the best; but we left out of our calculations the fact that Mr. Siddons and Mr. Godfrey and others of his pallbearers were not yet born, and, confirmed old heretic that I am, I am not sure that there is not some life in the old dog yet. (Laughter.) I observe from the agenda the immediate object of the Association was to induce all conductors of examinations to frame their questions without reference to any particular text-book; and further, those assembled at the meeting in 1871 undertook to use their best endeavours to increase the number of members, and to extend the influence of the Association. I am glad to believe that at this time when Canon Wilson now once more appears amongst us, he will find this programme has been largely carried out. We read in Deuteronomy of the patriarch Moses how, after forty years of wandering in the wilderness, his eye was not dimmed nor his natural force abated. I am glad that after our forty years of wandering in the somewhat arid regions of geometry, we may say the same of my good friend Canon Wilson. (Applause.)

Prof Hobson: I think Canon Wilson has brought out a most interesting point in the question whether we are to regard the statement on the parallels as an axiom or a postulate. It is very important to notice that in these early manuscripts it was regarded, with others, as a postulate, and not as an axiom, and for this reason, that modern investigations have established fully the fact that they are postulates and not axioms. The statement is not an axiom, because a consistent system of geometry can be developed which is just as logical as that of Euclid, and in which that axiom is not true. We have that for example in the so-called elliptic geometry. To say that two straight lines do not enclose a space is equivalent to the assumption that two straight lines do not intersect each other in more than one point. That again is not true in certain modern geometries which have been developed. An illustration would be the ordinary plane geometry on a spherical surface. Two straight lines, which on a spherical surface would be two great circles, do intersect twice,

and therefore do enclose a space. The point I want to bring out is that, in accordance with modern views of geometry, these are certainly postulates of a particular kind of geometry; they are not axioms in the sense that we are obliged to regard them as true. They are not true in certain other geometries which are just as logical as those of Euclid, and I think Canon Wilson is very much to be congratulated on having shown those who take an interest in modern research that these are very important historical points, and that Gauss, Bolyai, and Lobachewsky were not the first to discover them.

Prof. Harding: May I just add one or two words? I believe it is stated in all the histories, and I recollect very well that De Morgan says, that the equivalent to the word axiom was not known to Euclid at all. It was invented by Proclus. Euclid did not give himself away, I may say, by stating that these geometrical statements were axioms.

Mr. Garstang: In the new edition of Euclid by Dr. Heath this statement with regard to parallels is the fifth postulate, and not an axiom. Dr. Heath has not translated from an Arabic manuscript but from a Greek. The Arabic manuscript was correct in arrangement according to the Greek; and modern criticism has established that the Greek was correct in every respect, and contained the statement about parallels as a postulate; so that it is the English translations that have been in error.

Prof. Harding: Is it not more correct that there were two streams of Greek manuscript? I have always understood that there was the stream of common opinions and the stream of the axioms as we know them; and a manuscript was found in the Vatican in the early part of the last century which brought out again to Europeans the old stream of the common opinions and the five or six postulates. But I believe there were two distinct streams from ancient times. I am not an authority. I quote merely from translations.

Prof. Hobson: I believe Playfair is responsible for a good deal of what we hold to be correct. He made a good many changes off his own bat which were supposed to be Euclid but which are not so.

Prof. Harding: Simson made changes and did not admit that they were changes. Playfair acknowledged the changes he made.

Mr. Garstang: It is not really surprising that Euclid knew his own business, considering that he constructed the Fifth book. I refer that point to Prof. Hobson.

Prof. Hobson: I should like to ask whether, in the manuscript which contains merely a list of the propositions, the theorem of Pythagoras is amongst them.

Canon Wilson: These fragments of Adelhard's Euclid at Worcester contain only the definitions, the postulates, the axioms, and the first four propositions; sufficient to identify them with Euclid. There are other fragments of Gerbert's geometry found at various other libraries, those of Vienna and Paris for instance, and by piecing these fragments together a certain learned Austrian has reproduced Gerbert's first conception of geometrical facts as he arranged them.

The Chairman: We have had a very good illustration of how much "life there is in the old dog yet," judging by the delightful paper which Canon Wilson has given us. I do not know whether Canon Wilson is well aware that the magnificent edition of Euclid by Sir Thomas Heath which has been referred to is by one of his old pupils, who was an exact contemporary of my own, and who has done some splendid work. He has also paid much attention in the last few months to Aristarchus, who first suggested the heliocentric system. We have not, unfortunately, any actual treatise to this effect by Aristarchus himself, but his suggestions are well known from references in Archimedes. The only work by Aristarchus himself which has come down to us is an early one on the distances and sizes of the sun and planets, and this has been translated by Sir Thomas Heath, and I hope we shall soon have the benefit of his work.

When Canon Wilson challenged me vigorously as to my opinions, I remembered a meeting of the British Association at which Lord Kelvin was dipping into some fundamental questions. Prof. Fitzgerald and Dr. Johnstone Stoney (Fitzgerald's uncle, and a man to whom he owed his early training) were vice-presidents, and Lord Kelvin interrupted his own remarks by putting a question to both, and extracted contradictory answers from them. (Laughter.) I was wondering whether when my successor (Prof. Hobson) got on his legs, he and I might similarly be found to have contradictory opinions; but I am glad to say that that was not the case. We have had a most interesting discussion. I am sure you will all join in a most hearty vote of thanks to Canon Wilson, and also to Mr. Bushell whom we are so glad to see here. I hope Mr. Bushell will allow the agenda of the first discussion to be circulated in the *Gazette*. (Hear, hear.)

Mr. Bushell: With pleasure, sir; * and you may be glad to know that in those days we heckled Canon Wilson in just the way that we have heckled our Chairman to-day.

A hearty vote of thanks was unanimously accorded to Canon Wilson.

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* Mr. Bushell handed his copy of the first circular to the Chairman for printing and it will be found on p. 1 of this number of the *Gazette*.

THE ARITHMETIC SYLLABUS IN SECONDARY SCHOOLS.

By Mr. C. V. Durell.

The last ten years have witnessed many changes in the methods employed in the teaching of arithmetic, which to a very considerable degree are the direct product of the activity and exertions of this Association. Two signal benefits may be mentioned; firstly, uniformity of method in elementary processes has been secured; and, secondly, the educational value of practical applications to mensuration, physics, etc., has received due recognition. The object of this paper is to raise for discussion a question of quite a different character.

The tendency of recent years has been wholly in the direction of broadening the course read by the average boy. It is generally admitted that a state of affairs in which the mathematical programme of the majority of boys is limited to arithmetic, algebra up to quadratics or progressions, and the substance of three or four books of Euclid, is highly unsatisfactory: for this is equivalent to teaching the use of certain complicated tools without ever arriving at the stage where opportunities occur for practical application of knowledge so hardly acquired. The natural consequence is that, if we set aside the value of the training in itself, the years of mathematical study bear no fruit and contribute nothing to intellectual outlook in after life. If this is indeed true, and there is abundant evidence to support it, it is of the utmost importance to search for a remedy. The explanation, at least, is a simple one. The task of taking the boy of average ability through a systematic and comprehensive course of arithmetic, then drilling him in the art of manipulating complicated algebraical expressions, and instilling into him the rudiments of geometry, absorbs all the time that is at our disposal, and there is no margin for introducing him to the practical applications furnished by trigonometry, mechanics and the calculus. Our energies are directed towards laying a firm, substantial, and unduly ponderous foundation, and our resources are exhausted before we start upon the superstructure, so that in general the work is left so incomplete as to deprive it of all utility. The fact of the matter is that it requires a definitely mathematical bent of mind to be capable of assimilating the first stages of the work sufficiently rapidly to leave time for extensions. Now, if it were true that these higher branches are accessible only to the specialist, or if a thorough knowledge of elementary form was an essential condition for applied subjects, it would be useless to attempt to evolve any solution of the difficulty from a consideration of the character of the curriculum. But, fortunately, it is far otherwise. A numerical course of trigonometry up to the solution of triangles, excluding formal identities, requires a far lower degree of mental capacity than the ordinary double percentage type of question. A numerical course of mechanics, including the triangle of forces, moments, centre of gravity and machines, introduced by experimental work, offers less difficulty to the ordinary boy than does such work as the factorisation of a^2-a-b^2-b. The mental calibre for evaluating the greatest coefficient in the expansion of $(3x-4y)^{20}$ is far greater than is required for an elementary numerical course of the calculus, limited to operations on simple functions, and which affords a wonderful variety of problems and applications. The specialist, of course, works as readily in symbols as with numbers, and consequently those subjects which were formerly regarded as his special property were treated symbolically, and therefore assumed a form which could mean but little to the ordinary boy. But now that it is recognised that a more concrete presentation is not only possible, but also more illuminating, there is no

other obstacle than the mere fact of time, which closes the approach to these subjects.

It has been suggested that much time will be saved by eliminating from the programme of the amateur—as opposed to the specialist—a considerable section of algebraic manipulation, which is of little use in applications in more advanced work when treated practically or numerically. It is, of course, essential that the specialist should possess a high degree of dexterity in manipulative work, but the amateur requires rather a knowledge of principles. That this discrimination will save much badly-needed time is evident. The precise mode is the subject of discussion later in the afternoon, and it would be out of place to comment on it in this paper. My object is rather to call attention to another quarter where an economy might be properly practised.

The ordinary treatment of the subject of arithmetic, as evidenced by any of the text-books which have been published in the last ten years, may be classified under four main heads:

(1) Manipulation.

(2) Fundamental methods and ideas.

(3) Practical applications to mensuration and physics.

(4) Specialised problems.

These four groups are not, of course, in a strict sense, independent of each other; exercise in one arises naturally in the execution of the others. But if it is necessary to enquire what object is aimed at in the construction of a syllabus of arithmetic, it is clear that these are the four dominating constituents, and the final plan adopted will be a measure of the relative importance attributed to these sub-divisions.

As regards the first stage—manipulation—it is quite certain that until a boy has attained a reasonable certainty of performing accurately the ordinary operations of addition, subtraction, multiplication, and division for whole numbers, fractions and decimals, real progress in any applied branch whatever is impossible. Constant practice in these operations is essential at the initial stage, but it is open to question how far it is worth while spending time in explaining the "why" of fundamental processes or in drafting the work in a form which, although possibly more cumbersome, yet exhibits in a clearer light the underlying principles. It is no doubt best, in the first instance, to explain the reason for any given rule, but in purely manipulative work the main object is to secure accuracy and speed in execution. Moreover, it is an extravagance of time to introduce any manipulation which will not afterwards be of real practical value.

On this account it is fair to urge that too much attention is given to the application of prime factors to H.C.F., L.C.M., square root, cube root, etc. It is of real practical value for a boy to be conversant with the elementary principles of factorisation and index notation, but in the case of large numbers, excluding the specially constructed artificial numbers whose factors are limited to 2, 3, 5, 7, 11, it is surely a waste of time to employ such methods. Again, in practical applications where approximate answers are required, the use of logarithms saves time, renders accuracy more probable, and gives the answer correct to as many figures as the data justify: consequently it is not worth while to devote time to enforcing elaborate rules for contracted multiplication and division, although the idea of contraction, based on rough and ready methods which each boy devises for himself, is of real educational value. Similarly, in the case of square root, for practical purposes four figures are usually sufficient, and therefore the use of square root tables should take the place of the ordinary method of extraction, to which a single hour's attention would then be adequate.

The second heading includes simple areas and volumes, unitary method, proportion, ratio, percentage, variation, and possibly the most elementary of simple interest questions, where the interest itself only is asked for. In each of these cases the important thing is to ensure that the student understands the main principles of the subject; and he should not be required to attack such extensions and intricate applications as are usually associated with them. For example, if a class is to be drilled to solve the double form of question in which some profit and loss problems are cast, the time which must be spent is out of all proportion to the benefit received.

The practical work, mentioned in the third stage, requires no comment. The profitless elements of most subjects are the result of standardisation, and the introduction of this feature is too recent to admit of its having become stereotyped. Besides giving the student a broader outlook, and fitting him with an equipment that will prove of permanent value, this work also involves an amount of manipulation which impresses him with the real importance of accuracy and methodical arrangement. Lastly, we are left with that vast and ponderous mass of specialised arithmetical problems which still occupy so large a place in our text-books. On the one hand, there is that misnamed subject commercial arithmetic, which comprises discount, present worth, compound interest, stocks and shares. Has not the time come to obliterate these subjects from our curriculum? The argument that this form of knowledge, as tested in examination papers, is useful in after life will not bear investigation—the average man is not called upon to compute practical discounts or compound interests, and the professional does it entirely from tables; it is true that the meaning of stocks is of some utility, but this is a form of knowledge which is a simple matter for a mature mind, but presents great difficulty to the boy who is not old enough to appreciate their purpose; and further, such information as may be useful is a very small matter in comparison with the unnatural conundrums propounded in the stock exchange of the examination room.

On the other hand, there is a wide class of problems which have acquired a wholly unwarrantable importance owing to the natural conservatism of the examination system, to the narrow limits within which questions set in elementary papers must fall, and to the fact that examiners are compelled to set papers which approximate closely in character and quality to those of preceding years. As soon as it becomes necessary or customary to teach classes how to solve problems about taps filling baths or the relative motion of the two hands of a clock, it is certain that such questions have lost what value they possessed before they became stereotyped. There is, indeed, a strong case for the assertion that hard arithmetical problems, although drafted in a novel form, are unsuitable for a school course. Their educational value is directly proportional to the concentration of thought required; and it, therefore, follows that the maturity of mind of those students, who can grapple with them, must be such as to fit them for more advanced, more interesting, and more stimulating work. This is, however, irrelevant to the main issue. The examination system has led to an undue expenditure of time on specialised problems. This is inevitable if arithmetic is to be regarded as a substantial and self-contained subject for examination purposes. It is necessary to recognise the fact that a considerable amount of sound educational arithmetical work is quite unsuitable for examination purposes. The best results are obtained, and ideas suggested and assimilated, by a lightness of touch which is invaluable in the class-room, but useless in the examination hall, where method, systematic treatment, and thoroughness of investigation will alone yield high marks. It has been suggested that, although the standardised unreal commercial arithmetic is of little value, yet a training in the affairs of the world, civic finance, etc., is so valuable that its omission from a school course would be regrettable. This is very true; but why should it not be retained in the teaching curriculum but

deleted from the examination syllabus? In this way, liberty will be secured for the teacher to take the subject in the way that suits him best; whereas, if he is faced with an examination test, he is forced to treat it on standard lines, and the value of his individuality is destroyed.

Now the practical obstacle which faces those who would cut down the arithmetic course to the dimensions just indicated, is the excessive standard in this subject demanded by most examining bodies. As long as the various boards who control the Local and Certificate Examinations, Responsions, Army Qualifying, etc., continue to set questions involving a prolonged study of arithmetic, it will be necessary to retain a systematic treatment of the appropriate methods of solution in the curriculum. But if it is felt that reform is genuinely needed in this direction, then a united effort should be made to influence the existing régime. Now it is clear that it is neither possible nor desirable to omit a large section of arithmetic unless there is some alternative to offer, something which would better repay the time allotted to it and which would lead to more valuable results. There are various suggestions which may be made—an earlier study of practical solid geometry, or an elementary course of analytical geometry which might naturally be evolved from the graphical work in algebra and might be supplemented by a simple treatment of gradients of curves without recourse to the symbols of the calculus, or a programme of numerical trigonometry. As a starting-point, the last of these is probably the most practical and useful alternative. By introducing trigonometry at an earlier period, a large number of students who in the past have never reached this subject will obtain a working knowledge of its principles and will benefit in every way by the additional interest and power resulting from a course of this character. But it is reasonable to expect that, with the aid of the time saved by curtailing arithmetic, it will be possible to include in the course taken by the average boy not merely the practical applications of trigonometry, but also some introductory course on the calculus, of the kind indicated at considerable length in Mr. Mercer's recent book, and a straightforward and largely experimental treatment of mechanics. There are substantial grounds for this belief. In schools where it is already customary to have a course of numerical trigonometry at an early stage, the results have been highly satisfactory. Those who have examined the Austrian syllabus, recently published by the Board of Education, will have noted that it has been found possible to enlarge the curriculum of the ordinary boy in a remarkable manner. What is regarded as revolutionary in this country has there received official sanction and adoption. It would be beyond the scope of this paper to examine the details of that highly interesting document; it is sufficient to mention in passing that this syllabus, which is a *working* syllabus, cuts down the study of arithmetic to even smaller dimensions than I have suggested, omits from algebra the complicated work which has no direct application to future progress, and substitutes courses of analytical geometry, practical solid geometry, calculus and numerical trigonometry. There is, therefore, every reason to believe that the suggestion to enrich the curriculum is essentially practicable. What is required in the first instance for this change is that examining bodies should limit their arithmetic papers to manipulation and extremely simple applications, and should replace the customary second half of the paper by questions on numerical trigonometry up to the solution of triangles. This would afford scope for a wide group of interesting problems. Or it might be more desirable to suggest that a trigonometry paper of this kind should be accepted as an *alternative* to the more advanced portions of arithmetic. It is probable that each teacher produces the best results, from a broad educational point of view, when he is taking that form of subject in which he himself believes most: those, therefore, who hold that the second half of the arithmetic offers a better training-ground than other subjects already mentioned would still be able to retain

it. But those who believe in a more varied programme would then be at liberty to adopt it. The main purpose of this contention is to secure liberty of action. There are many who feel that the existing over-weighted arithmetic is an intolerable burden which cramps movement and fetters real progress. May it not well be the case that our gigantic syllabus is a survival from those days when the outlook was narrower, the topics less practical, and the treatment less elastic than is the custom to-day?

DISCUSSION ON THE ARITHMETIC SYLLABUS IN SECONDARY SCHOOLS.

Mr. Durell having read the foregoing paper,

The Chairman said: Nothing is more impressive than the anxious consideration which is being given by all teachers of mathematics with a view to saving time for this or that useful purpose. The time has been cut down by other subjects, partly by head masters and others, and it is now almost as scarce a commodity as water in the desert. It must be realized by all concerned that we are on short allowance, and that ordinary standards, such as are apt to be exacted by examiners, no longer apply. I remember one of my Egyptian friends telling me of a voyage in the desert during which he threw away an old marmalade pot, and one of the careful Arabs, who didn't like to waste anything, put it into the bag that contained my friend's night apparel, so that when he retired he found himself sticky with marmalade. It would be absurd to expect from that gentleman, who had only sand to wash himself with, the standard of cleanliness which we who have plenty of baths and water can attain to. (Laughter.) Mathematical teachers are rather like that. They have to satisfy examiners whose standard is too high for the available supply of time. But I hope we shall not feel the same lack of time in our discussion of the very able and interesting paper which Mr. Durell has given us.

Prof. Hudson: It seems to me that a great deal of the time which Mr. Durell wants might be saved if the teacher would not prepare his pupils for the examinations. All that is wanted in arithmetic is to teach the principles and the best methods of manipulation, and then the pupil ought to be able to do any sum the meaning of which he can understand. The difficulty of questions on stocks is that they use a language that the ordinary schoolboy does not understand. The boy finds a great difficulty in acquiring the meaning of "pounds" there, and he meets other difficulties of a like nature. It seems to me a vicious style of teaching to frame our work on questions concerning running taps and examples of that kind because such questions happen to have been set the year before. But the boy or girl ought to be expected to answer any reasonable question which is set in a

language that is understood. It injures the education of the pupil if he thinks he is to be taught a special way of treating questions of each particular sort.

This afternoon we have been referred to the early history of this Association, and the reference began with geometry. I know one of the original members, also greatly interested in the teaching of arithmetic, who took pride in setting an arithmetic paper which did not contain any £ s. d. His illustrations were all drawn from different sources. Now, at that date, perhaps even more than at the present time, what is called commercial arithmetic was well to the fore. Mr. Durell has mentioned "alligation." I remember it as a schoolboy, but I do not think I have heard of it since. I think that it is quite out of date. What is called "preparing" for examinations sometimes simply means trying to get the pupils to pass them without their understanding the subject. Let the pupils understand the principles of arithmetic, let them understand the language in which the questions are framed, and then they ought to be able to answer the paper. At the same time there are questions which do require a great deal of concentration and which are set in intelligible language. Such questions should not be set to very young boys, but, for the higher examinations, they do distinguish between the person who is capable of concentrated and prolonged thought and those who are not; and for this reason I think they are not to be discredited.

I have spoken at once, because at these meetings it is perhaps just as well to have a little opposition and to hear both sides, and I have dwelt upon matters which I disagree with rather than those I agree with. I was particularly pleased with Mr. Durell's remark that the pupil needed to understand the principles, and also with his references to manipulation. Perhaps I may refer to the latter. The best method is not a definite thing, but it is that method which is most suitable for the pupil in the stage of knowledge in which he happens to be. What is wrong is to adopt any method, however much it may seem to simplify the process in the beginning, which will impede the pupil's later progress. The teacher of arithmetic has got to remember, or at all events to hope, that he or somebody after him is going to teach algebra, trigonometry, and so forth; and therefore he should not adopt any method in arithmetic which will impede progress in these subsequent subjects. Notably, there is the old method of subtraction, which impedes the improved method of division. The difficulty of teaching the new method is that we are all habituated to the old; but our pupils are not habituated to it—those of us who are fortunate enough to have pupils who have never learnt any arithmetic before. (Laughter.) It is difficult to adopt a new manipulative method. I think it is the

duty of teachers to face and overcome this difficulty for the sake of their pupils. I think we want our pupils not to become acquainted with the old, but from the start to use the improved method. That will make future progress easier. Another weakness, to my mind, is that of pushing pupils too fast at the very beginning. We all of us know, I think, the difficulty of addition. Probably all would confess that if they had to add up a long column of figures they dare not rely upon their first result. I think it is true that, that is because we were all pushed on to multiplication too soon—before we had accomplished addition. If we avoid this early haste, I think we shall get rid of a great many of the manipulative difficulties which occur in the later stages.

Mr. Bushell: As long as examinations last, how does Prof. Hudson propose to persuade the ordinary teacher not to prepare for them?

Mr. Jackson: We shall all agree with a great deal that Mr. Durell has said. No doubt the traditional "financial arithmetic" is very unsatisfactory, but the specific suggestion he has made of throwing all financial arithmetic overboard requires consideration. I should like to tell the Association a little story which shows that there is another side to the question: There was a rich man who thought it would be nice if his wife had a banking account of her own. He gave her a start with a bank account and a sum of money. After about six weeks he had a note from the manager stating that her account was overdrawn. He said to her, "Well, so you've spent all your money." She replied, producing her cheque book, "Oh! No! I have got ten cheques left." That is a true story. (Laughter.) I think it has some bearing on this case. Financial arithmetic, even as it is now, may do something to give the school children, girls in particular, some insight into matters of business of which they ought to know something. This remark, which was made to me by Mr. Paget, entirely confirms my own impression. I think, therefore, that this side of the question ought not to be lost sight of. Ought we to throw the whole of financial arithmetic overboard because we all realize that it is rather silly to ask a child, for instance, what would be the cost of purchasing £25,000 of the stock of the Governor and Company of the Bank of England. I think we ought to consider whether that is the kind of financial education which is good; whether questions involving real comprehension of rates and taxes, insurance, municipal loans, banking, and so forth are not more a profitable form of training. I think that, for instance, a great deal might be learnt by discussing a case like this in its various bearings. Recently, in the north of England, it came to be known that there was coal underneath a field. How was this knowledge paid for? To raise this coal meant consider-

able expenditure, and before one single ton was brought to the surface, no less than £130,000 was spent on machinery. How was this money provided? The doubt is therefore suggested whether some development of financial arithmetic in the directions indicated would not be more desirable than its complete root and branch destruction.

Dr. T. P. Nunn: I should like to associate myself very directly with what Mr. Jackson has just said. I agree with him cordially in welcoming Mr. Durell's paper, and in his acceptance of the main thesis of the paper. At the same time I share his fear that in entirely discarding financial arithmetic, we should throw away an excellent thing. I do not propose to repeat Mr. Jackson's arguments, but merely to illustrate them. In an elementary school connected with this college we have been attempting to put into practice the views which he has enunciated so clearly. During this last term the subject-matter of at least one arithmetic lesson in the week has been the question of rates and taxes in various localities in and about London. Boys have been encouraged to bring information of their own on these points, and an interesting inquiry has been pursued as to where these mysterious rates go when they are surrendered in answer to the tax-collector's demand. The work was at first undertaken with reluctance by the student-teachers in charge of it, but they, as well as their pupils, have found it a fascinating occupation. We have very interesting graphical analyses which the boys have made, illustrating the different destinations of different parts of the rates. The contribution to the County Council has been followed up in much the same way. At a later stage in the syllabus, we have pursued the inquiries which Mr. Jackson has indicated. For example, the question of banking has been before us. Here I should like to express entire agreement with Mr. Durell's opinion, that financial arithmetic has been enormously over-elaborated. The function in the school of a subject such as banking should not be to teach complicated arithmetical manipulations; the object should be to get the pupils to understand the essential aspects of an extremely important phenomenon in social life. Consequently, great care should be taken to put the arithmetical work in a real setting. I may mention that in order to give reality to his lessons on banking one of our students, who was taking this work last term, selected the Birkbeck Bank for detailed description. Precisely at the moment required by his exposition, the Birkbeck Bank was good enough to be confronted with temporary difficulties. Events were followed by the class with intense interest until the bank had emerged safely from the crisis. We felt bound to acknowledge that it had served us a very good turn. (Laughter.) I think that work done in the spirit I have indicated supplies to the curriculum an element of real importance.

Thus in spite of a great desire to see Mr. Durell's plan effectively in operation in all schools, I hope that the success of his views will not render work of the kind to which I have referred impossible.

Mr. Garstang: I think we may all agree with Dr. Nunn and Mr. Jackson, and yet, at the same time, with Mr. Durell. Would it not be better if such work as has been advocated by Dr. Nunn were put into a course of domestic science, which is just as much required by boys as by girls? We should not then have the danger of over-elaboration. The great vice of these so-called commercial questions in arithmetic is this, that pupils are asked to deal with practical questions when they are entirely without practical experience. What we are advocating instead of questions bringing in principles and figures in this over-elaborated way, is that the questions put before pupils should bear upon their own experience, namely, questions in physics, or mechanics, or chemistry, and subjects of that kind; so that the pupils can go to their laboratory or elsewhere in the school building, and actually meet with the experience which is referred to in the questions. Without practical experience no child at school can be expected to deal in any competent manner with questions involving large amounts of stocks and things of that kind. All these questions on stocks and shares emphasize a wrong aspect. It is very easy to set questions, and ask how much stock at four per cent. can you buy up and sell out with such and such a result, and so forth; but the great question, after all, to the practical man is: Is this stock safe in which I am proposing to invest? It is not an arithmetical question at all which he has to solve. It is a question of the soundness of the concern. So that all this arithmetical work, so much over-emphasized in examination papers, and which takes an appalling amount of time to teach at school, is absolutely out of place. Its origin is to be found in Old Cocker. Cocker was a writing master in London, and taught arithmetic to the schools of his time. He based his teaching on the little points of business that the clerks were familiar with. His book went into something like sixty-seven editions, and every person who has written an arithmetic since has consciously or unconsciously taken some of his work from him and from other people that have followed him. I think our teaching should be more in accordance with the primary needs of the pupils in the schools. I am sure the Association must be much indebted to Mr. Durell for bringing these points so forcibly before it. I do not think we need fear at all that the interests of commercial arithmetic will suffer, because, in the immense number of hours of time we shall gain if these undesirable examples can be swept away, we shall be able to deal with economic matters which everybody will be glad to teach.

Mr. Grant: I am quite willing to admit that parents as a whole are a somewhat irresponsible body of people, to whom we ought not to knuckle under too much, but I think we ought to regard the view of parents a little. I wonder if the Association is aware that the only school subject of which the parent approves at the present day is arithmetic. The modern parent wants to know what is the use of Latin, Greek, and so forth, but Arithmetic, so far, has been his one stand-by. The position of some of us is going to be extremely awkward if you are going to sweep away these little problems in commercial arithmetic from under our feet. A little arithmetic of a commercial character is by no means a disadvantage to a member of the British Empire at the present day, and I should like to suggest a very simple remedy for giving ourselves time even for studying commercial arithmetic. If the Mathematical Association can induce the Government to adopt the metric system as a whole, we shall no longer be teaching two systems, but only one, and we shall then have time enough for most things. (Applause.)

Mr. Siddons: There are one or two things which I should like to say about this question. I have felt that it is really a question which perhaps the teaching committee ought to take up. There is no doubt that the present state of things is not satisfactory. Those who advocate commercial arithmetic, or domestic economy, or other things, do not want the existing text-books to be followed; and I think it would be a good thing for the committee to consider that matter and see whether anything can be done. The opinions expressed on the subject to-day will be extremely valuable to the committee, and I propose, at its first meeting, to suggest that we consider the matter in the course of the year.

One or two things occurred to me at the time Mr. Durell was speaking. He spoke about the importance of getting mechanical manipulation, of getting speed and accuracy. It seems to me that in the matter of manipulation the whole art of teaching is this, that we ought to make a child understand the processes which he is using; then, when he understands these processes, they must become rapid and accurate. The whole question, to my mind, is where shall we change from one aspect to the other. I do not think I have ever come across teachers who would not agree with me in saying that the line should be crossed, but can we agree where that line should come? That is the real difficulty. I think that is a point which the teaching committee might consider.

Then, again, Mr. Durell referred to the question of contraction. I believe the report on arithmetic which we wrote about eight years ago advocates a great deal of time being spent on contracted methods. I confess that I have tried to teach contracted methods. I used them myself at school, but I have never used them since —until I began to teach. I think Mr. Durell hit the nail on the

head when he said, what we want is the idea of what underlies the contraction.

With regard to the problems again, there is no doubt as to the way in which some types of these have grown up. I do not think they are only due to Cocker. They have come because some ingenious examiner has invented a new question. No doubt, first of all we had the bath with two taps, then a leak was discovered, and so they went on. There is not the slightest reason why we should regard them as important processes in themselves, but people will teach them for examination purposes.

With regard to commercial arithmetic, I quite agree, and I think Mr. Durell would agree, that we do not want to cut out all commercial arithmetic. Possibly Mr. Durell would like to go further in that matter than I should; but I am sure that we teach a great deal that is unreasonable and impracticable there. We ought to consider what can be done. It is no doubt the thing which the parent considers useful, and it can be made very interesting, and, I think, very instructive. But there is also a great deal which I am certain a parent would not consider useful, and on that ground I think there is the possibility of bringing about great changes. I mean to press the matter at the first meeting of the committee, and if people have got suggestions to make, I will take note of them, and lay them before that meeting.

Mr. Godfrey: In regard to the question of prime factors we shall notice, if we scrutinize the ordinary text-books in arithmetic, that this subject seems to be divorced from the main course that follows it. Prime factors have a good deal of time spent on them, and then, after very slight application to square root and so forth, they are dropped and nothing more is heard of the subject. In fact, their work seems to have been done. I am very much inclined to think that here we have a rather interesting historical survival. The arithmetic of the Greek philosophers was mainly concerned with the pure theory of numbers. Such matters as prime factors, perfect numbers, excessive, defective, and amicable numbers, to which certain almost mystical properties were ascribed, formed the type of arithmetic that interested the Greek mind; while the type of arithmetic that was associated with the uses of commerce was to the Greek philosopher of no account. The pure arithmetic connected with factors and the rest of it was still, I believe, the only part of arithmetic that interested philosophers right up to the Middle Ages. It was the only type of arithmetic that interested the philosophers of the Roman empire. Boethius wrote a standard work about this, and right up to the Middle Ages this Boethian arithmetic was taught by the schoolmen. They did not take any interest in commercial arithmetic; commercial operations and calculations were not

performed by them. The commercial arithmeticians were a different set of men. What happened after that, of course, was that the new system of notation came in from the East, through Italy; and then you had, side by side with the old Boethian schools of arithmetic, the practical schools whose operations were concerned with commerce. But there was still for a long time a division between the two types of schools. I am very much inclined to suspect that, in this lack of fusion of pure and applied arithmetic in present-day text-books, we do see a survival of that distinction between the pure arithmetic of the philosophers and the commercial arithmetic of the shopmen. We have arrived at the stage, I think, when we may well consider whether we cannot save time by a wise discrimination, and by the omission of such parts of pure arithmetic as do not lend themselves to real applications.

The Chairman: We have had the interests of the girls referred to, but we have not, so far, had the pleasure of hearing any of the ladies present speak. I will also venture to say that, some time before we tire him out, we should like to hear Canon Wilson speak in regard to the way in which we conduct our discussions in these modern times.

Mr. Abbott: I wish to raise one point which has been suggested by the remarks of Dr. Nunn and Mr. Jackson, who referred to the rather sweeping assertion of Mr. Durell that he would clear out all commercial arithmetic from the course. That, I think, suggests a danger which we ought carefully to guard against. As one who would like to reckon himself as being amongst the reformers, I am heartily in sympathy with the general principles of Mr. Durell's paper, but there is always a danger that the eager reformer may go too far, and err in the opposite direction. In these days there are some signs that mathematics is on its trial; that is to say, the position which mathematics takes in an ordinary school curriculum is somewhat in question. Head-masters and others are in doubt as to whether too much time is not spent on mathematics, and we have to be prepared with a case for our subject and to answer this important question: How far does mathematics play a part in the general education of the child? In other words: Does it play any part in the general culture of a child? Now, without attempting to define what is meant by culture, I would venture the opinion that one characteristic of a man of education and culture should be a capacity to take an intelligent interest in the phenomena and problem of his environment. Many of the problems which an ordinary person meets in everyday life are purely numerical and quantitative, and arithmetic does perhaps give a child more opportunity than any other subject of coming into contact with such phenomena. But there can be no question that the

arithmetic of the past has been divorced from everyday life. Now, what Mr. Durell objects to, what we all object to, is that at that particular point where the child is brought into touch with practical needs through commercial arithmetic the work has been too artificial; it is not that we object to teaching commercial arithmetic, but that what we have been teaching is something which is *not* commercial arithmetic. For example, we have talked of "True" Discounts, whereas "true" discount has never existed in everyday life. We have had plenty of other instances, some of them referred to to-day, of the kind of artificial examples which have grown up, partly as a result of having to prepare for examinations. The point which I should like to emphasize is that it is essential that we should mould our arithmetic to meet the needs of everyday life; and I think Dr. Nunn's illustration of the way in which it has been pressed into service in this very building affords an excellent example.

Miss Punnett: It seems a pity to leave unanswered the Chairman's invitation—or challenge—to someone to speak on behalf of the girls; so perhaps he will allow me to say a few words on the so-called commercial arithmetic in its relation to them. I was especially interested in the fact that, when Mr. Jackson wanted just now to lay before us a particularly bad instance of ignorance of such matters as banking, his example was taken, either by accident or by design, from the case of a lady. I am convinced—and I am sure other teachers of girls will agree with me—that not only is it most important that girls should be given such knowledge of arithmetic as will be of real use to them in solving the problems of daily life, but that the acquirement of this knowledge may be made—in fact essentially is—most interesting to them. In the Demonstration Schools attached to this College we are giving our girls towards the end of their school course arithmetic lessons which have this practical bearing on their lives outside the school. They are taught to keep simple household and personal accounts, not in the formal way in which certain arithmetic books treat that subject, but as a means of answering such questions as "If you were required to manage the expenditure of such and such an income for a family of so many for a week, how would you do it?" Such work, besides being valuable in itself, leads immediately to other problems of wider interest. For example, one of the items that must be included in every wise arrangement of expenditure is "savings." This leads at once to the question, "What will you do with your savings?" and so to the Post Office Savings Bank, and later to the banking of larger sums and the details connected with that process. Such exercises in keeping accounts—or rather in distribution of expenditure—and the treatment of the topics naturally arising out of them, prove not

at all too difficult for even the backward and non-mathematical girls. Indeed the only objection that has, so far, been made to this work is that the mathematics it involves is so simple that the exercise of it does not entail sufficient progress in "pure" mathematical knowledge. But surely such a course as I have tried to indicate, in addition to being extremely interesting and stimulating to the girls, supplies just that practical application of their mathematics without which we must all feel that the subject is for them barren and useless.

Mr. Butt: As we are discussing the usefulness of commercial arithmetic to girls, may I suggest that it is very important that young ladies should be taught something about "bucket-shops," because those young ladies who have a little money invested, and especially maiden ladies, are generally flooded with horrible papers which invite them to invest their money through these "bucket-shops." I know of a case where, quite recently, a lady, renowned for her charitable works, lost £14,000 by their means. It is a terrible thing that, in England, we cannot starve out such an iniquitous system. From time to time, I have spoken about the London University Examinations. If you will refer to what I said two years ago, you will see what I felt about the examiners giving such questions as those asking for the compound interest for 17 years, and so on, without giving any logarithms. Last year I was able to report a change, and I think I may say that I am perfectly satisfied with the September examination of last year. There was one rather difficult question at the end; but it was, on the whole, a very sensible paper. It is of great importance to us that these examiners should follow the proper lines, because we have to work for them. I mean to say that many of the teachers in Middlesex are whipped into line by them. It is useless for any one to suggest that we can put on one side any particular line of work. That would mean the failure of every pupil, and the failure of pupils means, in the long run, the failure of the school. Therefore we cannot avoid those things which are obnoxious to us. The only thing we can do is to train the pupils to have the sense to select that which is useful.

The Chairman: I am very much obliged to Mr. Durell for his paper. I may reply perhaps in two words to one of his questions. I can assure him that all the astronomical records made at the Observatory are checked and the records are "ticked," as it is called, on re-examination, or an entirely independent computation is made and the results afterwards compared. On one occasion I remember, at Greenwich, there was a good long bit of work to be done, and it gave me an opportunity of finding what was the average number of mistakes made by computers who had had a fair experience in arithmetic. What I am going to conclude may not apply to bank-clerks, but in ordinary computations

of a simple character we found that of eight computers six made about one mistake per hour; one made a mistake every half-hour, and the other made one mistake every two hours. My business now is to ask you to return your thanks to Mr. Durell. Would he like to make any reply to points which have been raised in the discussion?

Mr. Durell: I should like to ask whether the issues raised have been definitely referred to the Committee to deal with. It seems to me that some of the points under discussion are of pressing importance. The papers set in Responsions, for instance, are highly unsatisfactory in character, requiring as they do considerable exercise in particular types of questions, which are of no value afterwards.

On the motion of Mr. Siddons, seconded by Mr. Godfrey, a hearty vote of thanks was accorded to Mr. Durell for his paper, which, it was understood, would be developed into a report by the Committee.

A DISCUSSION ON THE REPORT OF THE COMMITTEE ON THE TEACHING OF ALGEBRA AND TRIGONOMETRY.

Mr. Siddons: First of all, in bringing the report before the Association, I should like to point out that it is going to be re-considered by the Committee and that criticisms offered on it to-day will be duly considered. A statement to that effect appears in the *Mathematical Gazette*, which ought to have been in your hands some days ago, had it not been unfortunately delayed.

With regard to the history of the report, it is the outcome of a paper read before the Association last year by Mr. Godfrey on the question of algebra; the main point of that paper was the possibility of dividing algebra into two parts, the educational and the technical. The educational part might be defined as the part which belongs to general education, and the technical part as that which belongs to the future mathematician, the future engineer, and to those who are going to make special use of mathematics in their professional career. That paper was referred to the Committee, and the Committee has drawn up this report.

I believe that all members of the Association have received a copy of the report, and I should like to give a brief outline of its contents. First of all it considers the aims of the teaching of algebra; secondly, it considers what parts of algebra should be included in the essential or educational course and what parts should be excluded from that course; and lastly, it expresses the hope that

examiners will fall into line with the report and help by having their papers set on the lines recommended—papers such as those for the Leaving Certificate, Responsions, Previous Examination, London Matriculation examinations, as well as those for the entrance examinations to the professions. You will see that we did not recommend that questions on what we classed as "non-essential" should be excluded, but merely that if such questions were set, then alternative questions should be set on other subjects so as to give liberty to teachers who thought they could get more value out of those other subjects than they could out of the part that we felt belonged to the course of the professional mathematician. A distinctive feature of this report is that it considers the aims of algebra teaching.

Now let us examine some possible criticisms. First of all, people may say that this will lead to a lower standard of mathematical work. Well, that is very definitely opposite to the wishes of the Committee; the Committee, in their report, definitely say that they are opposed to a lowering of the standard, and they suggest how that standard may be kept up. And if people have some freedom of choice in what they teach they are more likely to keep up the standard than if they are compelled by the pressure of examinations to teach subjects in which they do not believe. The second criticism—and I have no doubt this is the point round which most of criticism will centre—is upon what shall be included in the essential course and what shall be excluded from it. In considering this I would like to point out that everybody will be compelled to teach what appears in the essential course, so that if you wish to see a certain thing kept in the essential course it involves compulsion on all teachers. The third criticism is, that this report has been engineered by people who want more time for weighing and measuring, and to this I object very strongly indeed. In the first place, I have had something to do with drawing up this report, but I have not got that aim, and I do not like to feel that I have been used as a tool by other people to further their objects. Secondly, I deny that weighing and measuring are a waste of time. As an actual fact I believe that by the increase of interest to which they give rise they really effect a saving of time. My own experience in one particular division, where I have some weighing and measuring to do every term, is that arithmetic comes very much more quickly—I mean that when we included the practical work in our lesson I spent much less time on arithmetic and practical work than I should otherwise spend on arithmetic alone. For instance, I suppose many of us have to revise work such as the multiplication and division of decimals, and boys have got to be more accurate there, and if they are working at results which they have obtained from weighing and measuring they do their work much better

and more willingly, and fewer lessons are required than if they were set to do a page of dull sums out of a book.

I do not propose to refer again to the criticisms under the first and third headings, but I should like to consider some of the criticisms as to what shall be included in the essential course and what excluded from it. Some people will say, "Why cut out a lot of the heavy work, for example, complicated fractions?" Let us consider on what grounds such considerations are based. Is it that they produce accuracy and power of concentration? The question arises, Does this dull drill do what is claimed for it? If so, is it worth the time spent upon it? My own opinion is that these heavy examples do not do what is claimed for them, but they do produce nausea. It is a fallacy to suppose that accuracy and powers of concentration are best acquired by means of examples which are devoid of interest; whenever this supposition has been put to experimental test, it has proved itself false. Not only is it untrue that a faculty is better cultivated by setting it to produce useless results rather than results of obvious meaning and value, but there is also no reason to suppose that the facility which the mind gains in dealing with useless problems can be carried over in any very great degree to profitable fields of exercise. It follows that technical and formal exercises can be justified only by their relevance to work which has a substantial value.

In Adams' *Herbartian Psychology* we read, "The theory of interest does not propose to banish drudgery, but only to make drudgery tolerable by giving it an interest. We have seen that what is interesting is by no means necessarily pleasant, but it is something that impels us to exertion. The principle of interest braces the pupil up to endure all manner of drudgery and hard work." The report proposes to make drudgery tolerable by giving it a meaning, and it hopes to do this by passing on to subjects, such as numerical trigonometry, which the boy finds "meaningful," and by eliminating as far as possible merely symbolical and abstract manipulation, such as harder fractions, which the boy finds meaningless.

Again, what can be said for the rule for H.C.F. In using it a boy does not reason for himself, but merely reproduces a requisite number of copies of the reasoning of his betters. What can be more blighting than grinding at these H.C.F.'s, and how frequently they have occurred in Responsions lately!

As to Progressions, this being work which the Committee think lends itself to cram, they on that ground recommend that it be postponed and put into the advanced course. No doubt many people will still teach Progressions and teach them sensibly, as they have no examinations hanging over their heads; but we do not want to have them set in examinations so that

everybody will be compelled to bring their boys and girls up to a certain standard and get them through by cramming them.

Another point concerns the use of logarithms. We recommend that the use of logarithms should be included in the essential course. Now I think that is a place where perhaps the report will have to be altered slightly. Apparently there is some misunderstanding. It has had read into it, "That people shall be taught to use logarithms without having any explanation." Well, that certainly was not the meaning of the Committee. The Committee decidedly meant that there should be a certain amount of explanation.

Another thing that I must refer to is the subject of Functionality. "The term itself is enough to frighten any boy out of his wits." At least, that is what I am told. We do not mean anything terrible. We wish that, after a boy has measured the diameters of several circles and their circumferences, he should be led on to suggest that the circumference bears some relation to the diameter. It is not essential that we should use the word "Function," but simply that we should convey the idea that they are connected quantities, that the circumference works out in many cases to be 3·14 times the diameter, and then the boy will jump to the conclusion that this is the case in general. That type of work is, I think, making some use of the graphical idea. I believe people who are afraid of the word "Functionality" fear the term rather than what is meant by our recommendations. The idea is, we think, the most important that a boy has to acquire in his elementary mathematics. The report recommends that such work should replace the ordinary questions of Variation.

I hope we shall have many opinions expressed on this report. I feel it is very important we should have it discussed, and whatever questions are raised will be of help to the Committee in the re-consideration of the report.

Dr. Nunn: Like Mr. Siddons, I am cordially in favour of the proposals of the report. Speaking, of course, as a teacher and not as a member of the Committee, I should like to say that I welcome its recommendations because, if they were adopted by examining bodies, it would, for the first time, become possible to attack the problem of teaching algebra in the proper spirit. We all agree in these days in rejecting the notion that the boy is a receptacle into which information may be poured at the teacher's will, but some implications of that rejection are not, I think, recognized as clearly as they ought to be. If the boy cannot be considered as a passive partner in the process of instruction, it becomes necessary for us to determine the conditions under which his mind naturally works. Now, there are various motives to

intellectual activity, and I think that we can distinguish at any rate two of them as of primary importance and of great relevance to the matter before us. There is the pure scientific motive, the motive which is operative in the intellectual activity of the original mathematician, the professor at the University, and so on. In such a work as the *Principia Mathematica* of Mr. Russell and Dr. Whitehead you have the fullest expression of this motive. The aim of these authors is to arrange the whole body of mathematical truths in a perfect deductive system; that is, to exhibit it as derived in accordance with clearly-conceived principles from a small group of "axioms." I think there is no doubt that the study of mathematics, as it goes on in the ordinary class-room, was until lately supposed to be actuated by the pure scientific motive. The real meaning of the "dethronement of Euclid"—a revolution in which Mr. Godfrey and Mr. Siddons played a part to which they have modestly refrained from referring—was, that we decided that the pure scientific motive is not a motive which naturally sets boys' minds working in mathematics. If this is the case we must find another. In my own mind there is no doubt that it is the one which I have ventured to call the utilitarian motive. By giving it this name I do not mean that the ordinary school-boy keeps a look-out for things which will be useful to him in his after career and, following the famous advice of Captain Cuttle, makes a note of them. I mean simply that he respects mathematical and scientific truths not for their value as pure knowledge, but on account of their applications in matters of practical interest. Thus he will gladly learn how the sailor finds his latitude and longitude—information of all the more value in view of the romantic possibility of ship-wreck on a desert island. In the same way he will take the keenest interest in Archimedes' Principle if it is presented not as a "property of fluids," but in explanation of the raising of the "Gladiator" from the sands of the Solent. It appears to me that we have here the motive to which appeal must systematically be made. If so, there must be a radical change in the way in which mathematics is approached in the class-room. We must recognize the working of the utilitarian motive throughout, and must base our teaching upon its presence and efficacy. If you are not willing to admit this principle in algebra, I ask you to consider what other alternative is open. There is but one. It is to build up the subject as a self-contained science in the manner typified by Professor Chrystal's two-volume work. There is at least one recent text-book which deals with the teaching of elementary algebra upon these lines as well, I think, as it can be done. I have a great admiration for the ability and sincerity shewn in this book, but I cannot believe that permanently satisfactory results can attend teaching based upon the pure scientific motive.

Such teaching must be recognized as having broken down all along the line.

We must, then, abandon the idea of building up algebra as a pure science resting on a basis of axioms, and must present it as a useful tool, a means of analysis and description of the behaviour of tangible realities. This point of view may be made clearer by a simple illustration. The identity $a^2-b^2=(a+b)(a-b)$ has generally been taught simply as a formal truth, as an early step in a theory of functions. From the utilitarian standpoint it is rather to be regarded as a means of shortening certain computations. For example, let *ABCD* be the plan of a square courtyard, into the corner of which a square building *DEFG* projects. Calling the sides of the squares a and b, the area of the yard is a^2-b^2. But we observe that if *EF* is produced to meet *BC* in *H*, $FH=BH$. Thus, by cutting off *FC* and placing it so that *FH* fits on to *BH*, the figure is simplified into a rectangle $(a+b)$ by $(a-b)$. We conclude that $a^2-b^2=(a+b)(a-b)$ and recognize that this transformation enables us to shorten the labour of computing the area. It is easy to find many examples in which the same principle can be applied—the manipulation of the symbolism being merely a description of the results of manipulating the geometrical figures. But when we reach the case of a ring of external radius a and internal radius b, and ask whether the identity $a^2-b^2=(a+b)(a-b)$ can be used to shorten the computation of its area, the case is different. We can justify the application of the identity here only by shewing that it follows from the properties of the products of numbers and not merely from the properties of certain figures. In other words, "algebraic multiplication" comes in as a process which is to assure us of the wider validity of our identity. Now it appears to me that algebra should be taught to boys in school on some such lines as this example has indicated. The deductive side should be kept in subordination to the practical side. The central motive of the work should be the desire of practical or intellectual control over things that lie really within the compass of the boy's experience and interests. I believe that the whole of the subject can be developed on this basis, and that all arguments that can be brought against such a treatment fail like the arguments against teaching geometry on modern lines. They are met by asking the objector to produce his alternative method. The only legitimate alternative that he can produce is the pure scientific method which experience has abundantly proved not to be suitable for children. Moreover, the treatment which I advocate can, I believe, be defended on mathematical, as well as on pedagogical grounds. For example, if instead of teaching algebra as a mere manipulation of symbols based upon certain formal laws, you develop it as a means of dealing with realities, then you are

keeping your boys on a line which leads to what has been called the arithmetization of mathematics, a subject upon which our new president made some very illuminating remarks in his recent address to the British Association.

I should like to supplement Mr. Siddons' remarks upon Functionality. There can be little doubt that the idea of Functionality is one of the most important that a boy can carry away from the course of elementary algebra. It seems to me that the difficulty of teaching that idea has been greatly exaggerated. There are, after all, a very moderate number of forms of relation of great importance, and, as Mr. Siddons has pointed out, it is quite easy to give a boy a really practical working knowledge of these by the graphical method. The various typical curves serve as visual symbols of the abstract relations which the boy has studied in the concrete instances of the physical laboratory or every-day experience. They are readily recalled to memory, and enable the boy to keep his grip upon the symbolized relation and to distinguish it clearly from others. (Applause.)

Mr. F. W. Dobbs: I am afraid I am going to introduce a discordant note. But I realize the importance of the work that has been done by the old Association for the Improvement of Geometrical Teaching and the advantage of the changes since then, so I hope it will not be thought that anything I say is said in a cantankerous spirit. Perhaps I may first of all refer to Dr. Nunn's speech. If I felt that I could at all follow him, either in his views or in the extremely interesting way in which he puts things, then I think I should be able to agree very much more heartily with this report. Dr. Nunn's speech brought one leading doubt to my mind. Supposing there are masters in a large school able to follow the line of teaching which he recommends, yet continuity in the teaching of boys who pass from one master to another would be made very difficult; all the boys must be taught on the same lines, and therefore any root and branch change that is suggested ought to be allowed a considerable time to come in. That is my chief objection to the report as it stands; I think the measures are too extreme. I refer now to Mr. Siddons' statements. I think it would have been fairer if he had allowed those of us who have expressed opinions antagonistic to the report an opportunity of voicing such objections ourselves, instead of anticipating us by placing them before you himself. I want to say that I look forward to the time when all boys leaving school will have some working knowledge of trigonometry and also some knowledge of mechanics; the only thing, as I said before, is that I think the report is too extreme. I should like to make two remarks by way of preface. First, as the examinations affected are not competitive examinations, but only pass

examinations, the student we have got to consider is the average boy—or rather the dull boy; and in considering the educational value of algebra we must remember that there is first its utilitarian aspect, then there is the exercise of thought—the teaching of the intelligence—and then there is what I think is as important as either of those, especially in the case of the rather stupid boy, and that is the acquirement of the power of concentration, not only habits of concentration, but actually the power. During a lesson, it scarcely matters in what subject, it is difficult to tell whether a boy who has got his eyes on the book, and who is not actually working out the question, is attentive or not; and not only that, the boy does not himself entirely realize whether he is attentive. There has been a rumour lately that some headmasters, as a counterblast to what has been done in the attempt to abolish compulsory Greek, are talking of abolishing compulsory algebra. There was indeed a letter referring to this project by a friend of mine, the headmaster of Shrewsbury, in the *Times* a short time ago. I think the educational value of algebra, on which perhaps we all agree, is worth considering as a really important answer to that suggestion. I want to emphasize this point, that it is the habit of concentration that is of the utmost importance, and yet the report allows no sort of question by which it can be exercised. If those who have got a copy of the report will look at page 5, paragraph 5, they will see that in examinations we are allowed to set questions on fractions, so long as their denominators are easy, but that we are not allowed to set questions with more difficult denominators. In examinations we are allowed to ask questions on these factors, but we are not allowed to apply them to the simplest examples possible. What does that mean? I think it means this: to inculcate the habit of concentration, we want a certain number of questions in which a boy does not always have to grope in entirely new ideas. We want to exercise him in work in which the method is perfectly simple; always similar, but not the same; in which the reasoning goes on from line to line, so that his mind—I am only talking of the stupid boy—is thus always kept working. We are told in answer: you must still teach dull boys these questions, but why should you compel other boys to be taught them? I do not think that these questions, so long as we avoid "sky-scrapers" (we do not defend what has been called the "sky-scraper" form of fraction), are any serious drawback to the clever boy, or even to the average boy. Another point is this. I draw your attention to the fact that we are considering pass examinations. If these boys work (they have got to get into the University, the Army, the Navy, or whatever it may be), are you going to make the way easier for the intelligent but idle boy, or are you going to give a rather slow, hard-working boy an equal

chance? I have laid great stress on the idea of the stupid boy. I know another answer has been given. "Do not have him in the public school or in the university." That is rather like a solution to the political problem of what is going to happen in regard to the Irish question. The solution suggested is that all Irishmen are to go back to Ireland, and then the island is to be submerged. I am quite as willing to discuss one question as to discuss the other.

As to the details of the report, I want to suggest that you do need to make it possible in these pass examinations for a stupid boy to have some chance of showing that he has worked hard, and for that reason I should like to see harder fractions than those allowed, as well as such straightforward methods as simple cases of H.C.F., and easy questions on square root. The only other point of importance which I will mention is that which concerns "variation" and "functionality." I believe enormously in the subject of graphs; but why is variation suggested? How far, if this is a syllabus for examinations, how far are you going to succeed in getting a sufficient number of questions involving simple ideas, without finding after the first two years that a few artificial types have been standardized just like the "two taps and a leak" questions which were so mercilessly criticised just now.

When Professor Hobson thinks of a very stupid and dull pupil of his, he will perhaps realize why I am so much in sympathy with the dull and stupid boy.

Mr. Barnard: I did not expect to make a formal speech, so I am afraid that what I have to say will be rather fragmentary. In the first place, I would say that at Rugby the mathematical staff are conservative, and we do not altogether agree with this report. Our chief reason is that we consider the course laid down to be inadequate. We think that the average boy would get through the course laid down by the time he is sixteen. If he stays at school till he is nineteen, what are we to do with him?

We are expected to devote an extraordinary amount of time to weighing and measuring, and although our opinion is that these are most excellent things, yet we feel it much more desirable that they should be dealt with by the men of science in the physical laboratory. If instead of spending an extreme amount of time on these things, we take certain boys on to do the kind of work suggested in the report, such as the differential calculus, we should have to teach on entirely different lines those who are to learn enough algebra to tackle these things seriously. I think that would lead to great practical difficulties. I might say that all of us agree with what Mr. Dobbs said in a letter which he circulated, the main points of which I believe

he mentioned just now. Our idea is that a boy requires more drill than is considered desirable in the report, and I think, if the report were altered to make it pretty much on the lines which Mr. Dobbs suggested just now, we should be in favour of it. But I am bound to say that I should like to see one other addition. I should like to see it stated somewhere or other in the course of the report that the various parts of the subject should be taught in such an order as will admit of a logical development. It has always seemed to me to be most desirable that, whatever text-book is used, the statements contained in such a book should be absolutely sound. If one is not able to explain a thing completely, it is quite easy to leave out the explanation altogether, but I think whatever explanation is given in a text-book should be sound.

Mr. Daniell: I have had great difficulty in understanding what are the precise objections of the last speakers to the recommendations of the report. To take Mr. Dobbs' speech first, I find that he mentions the difficulty in regard to mathematics being taught continuously up the school, because the teaching is not all in the hands of one member of the staff. Surely for most members of the staff there must be a certain amount of limited specialization and a certain amount of general work; that is to say, in a large school we expect to find one mathematical specialist and certain other members of the staff who are sufficiently interested in mathematical teaching to be able to take it up as a sort of limited speciality and who desire to carry out reformed methods. The latter are likely to be interested in the teaching of mathematics from the professional desire to improve mental training. Reformed methods are likely to appeal to such men. But if it were not so, the argument that reform is not to be undertaken because you have not got the power of carrying out the reform by highly trained specialists throughout the school, is an argument which would stop all reform in every branch of the curriculum. I do not think we can allow it to weigh with us in the Mathematical Association. Then we are told that the report is too extreme. I have been listening carefully to find out where the report is too extreme; but I did not hear of any particular in which it had been found so. My own feeling as regards the report in detail is that I should agree with it entirely, except that I regret that Progressions have been ruled out of the compulsory list. We were told, however, that we have not only to deal with the difficulty of the staff, but that we have to deal, too, with the difficulty of the stupid average boy. Well, I do not believe in the stupid average teacher, and I do not believe in the stupid average boy; but I do believe that we are very often guilty of stupid average teaching. (Applause.) That is another matter, but to the average boy I deny the

charge of stupidity. If a boy is interested in a subject, it is astonishing how the stupidity clears away. Better educational methods will get over the difficulty of the "stupid average boy." I think Mr. Dobbs agreed with the report up to a point. Mr. Barnard, however, considered it inadequate. The course recommended is, apparently, too extensive for the first objector, and too limited for the other. It looks as though the authors of the report had been successful in striking a happy mean. I do agree with Mr. Barnard that there is a danger of too much weighing and measuring. I have done a great amount of that myself for twenty years, and I see that it can easily be overdone; and I certainly caution you against doing so much weighing and measuring as has been done by science masters generally within the last twelve or fifteen years.

May I now refer to the report as a whole. About a month ago I wrote to the Secretary, and pointed out that it was most important that the Mathematical Association should consider the proper place of mathematics as a branch of education in general —*that we should base our recommendations as to practice upon a previous consideration of the aims that we were trying to achieve as educationalists*; and I very warmly welcome this report for the innovation it makes in putting first the principles which should influence our teaching of mathematics. It places in front of every other consideration the question, "Why should we teach mathematics? What is our aim in mathematical education in the schools?" The position of mathematics in schools is being assailed; the suggestion has even been heard that we should make algebra an alternative to Greek! For my own part, I consider that mathematics is a most important branch of education, and that it must be considered an "outlook" subject. The mathematical equipment of boys—and, let me add, also of girls—should be such as to give them a better understanding of the world around them. If we approach the subject with the idea that our teaching is going to produce that effect, then we can lay down lines along which we can pursue our aim in detail. Now, Sir, I have merely to add that I very strongly agree with the recommendations in the report, except that, as regards Progressions, there seems to be too much fear of examinations and of examiners. I think there is a confusion between difficulty and high standard. An examination paper may be of a low standard and yet be difficult; it may be of a high standard and yet avoid difficulties of technique. I could mention recent examination papers in algebra which are difficult without being of a high standard. Complicated divisions and fractions do not require an advanced—or even a good—mathematical training, but do need a great deal of practice, if accuracy is to be obtained in the working of a long sum. We

rather want to set a higher standard of mathematical thinking, with less difficult problems. I know that the working of a large number of problems, similar but not the same, is a very useful device in the art of boy-minding, but we should look on the teaching of mathematics, as has been done in this report, from a higher standpoint than that of boy-minding or of girl-minding. (Applause.)

Prof. Lodge: I quite agree with most of this very valuable report, but I want to point out one or two things which seem to me to be not quite consistent. I refer to the fractions which, in page 5 of the report, are called "complicated fractions." We read: "More complicated fractions, such as

$$\frac{2x}{x^2+4x+3}-\frac{3}{x^2-1}\cdots,$$

should be classed as non-essential for elementary examinations." Then, at the botton of the same page we read that factorization of quadratic expressions should be mastered. Now, it seems to me that if the boys had learned to factorize, questions connected with fairly simple factors will always be liked by them. If that is so, it is surely rather a pity that fractions with quadratic denominators should be ruled out as too hard for a boy who is not specially mathematical. I quite understand that the point of view is that, instead of spending a long time over work of this kind, the boys should go on to trigonometry and easy mechanics. That would no doubt be the answer of the Committee to Mr. Dobbs and Mr. Barnard. But if the boys are doing factors it seems to me to be a very small thing to make this distinction in regard to quadratic denominators. Personally, I should like to see those fractions included within the limit of the elementary course. With regard to H.C.F., mentioned on page 6, number 7, if the principle that the H.C.F. of A and B is contained in $mA \pm nB$ is laid stress on, I know that boys, not necessarily very clever boys, will take hold of the idea very well indeed. If they were encouraged to use that principle, it seems to me that that would be a useful part of the easy philosophy of algebra and arithmetic, which it would be a pity to leave out of the things recommended in the report. Of course, the report is rather in a difficulty. Those responsible for it are thinking partly of what is good for the student to learn and partly of what they are afraid of seeing in the examination papers; in fact, I think Mr. Siddons said that we rather like the boys to learn the things, but we do not like to see the questions set at the examinations. I do not quite know why the distinction is made. I know there is the fear that, in the case of the boy who is just on the edge of passing an examination, two

or three questions of the sort might just turn the scale against him when otherwise his elementary knowledge would suffice. And this consideration puts the Committee in rather a difficulty. There are really two distinct kinds of questions to be decided, viz.: "What is essential for teaching, and what ought, or ought not, to appear in the examination papers." Both these questions ought to be borne in mind in revising the report.

Mr. Daniell: As the Committee are going to reconsider the report, may I add a suggestion in the hope of meeting the practical difficulty of examinations? The Committee have been anxious to keep down the amount submitted for examination, and to allow a larger syllabus actually to be taught than is to be set for the examinations. My suggestion is, that a number of schools should agree to put forward a certain common syllabus which serves for the main part of the examination, and each should also send a short additional one of its own work beyond the common minimum.

My experience as an examiner enables me to say that it is quite practicable to set a paper covering a large number of syllabuses combined in this manner. This method would leave the schools much liberty of choice in regard to what should be taught. If the Committee think this a good principle, they might see their way to include some suggestion of the kind in their recommendations.

Miss Punnett: I should like to say a few words in hearty support of the report from one point of view. I refer to the statement in the footnote on the first page, in which we are told that the word "boy" is intended to include "girl." I feel that we who are concerned with the teaching of girls ought to be cordially grateful to the Committee for putting them, in this matter, on the same footing as the boys,—for I am sure we may acquit the Committee of any intention of leaving a loop-hole for the admission of differences between boys and girls by the use of the phrase, "except where the context renders such an extension of meaning obviously unsuitable." It is not because I think that it has been proved that mathematics for girls should be exactly the same as for boys, that I think we should welcome this attitude on the part of the Committee. Many opinions have been expressed for and against the theory that boys and girls are equal in this matter—it is evident, in fact, that the time is not ripe for a definite and unanimous conclusion on this point. It is very difficult to say to what extent the present comparative inferiority of the mathematics in girls' schools is merely due to an ancient tradition to that effect. The best way of solving the problem would seem to be to put the mathematics for boys and girls, for the present, on the same footing and to be guided by experience in making the necessary modifications in the work

of the girls. The particular point on which I should like to say a few words is the subject of trigonometry. There seems, in certain directions, to be a tendency to assume that girls are not interested in such abstract matters as sines and cosines. It is indeed possible that further experience may show that if there is a difference between boys and girls it is that the average girl is somewhat less interested in the purely abstract side of the subject than in its concrete applications. Whether this be so or not, the point scarcely affects the suitability of trigonometry for girls; for numerical trigonometry, if approached from the practical side, is surely one of the more concrete forms of mathematics, and it seems the greatest pity in the world that it has been so common to omit it from the syllabus in girls' schools. I believe that problems in simple surveying—for example, so fascinating an operation as ascertaining the height of a tower or of a tree when it is possible to approach within several hundred yards of it—would prove at least as interesting to girls as to boys, and it is possible that the same kind of argument may apply to all the forms—though not necessarily all the details—of practical applications of mathematics suggested by the report. In any case it is most encouraging to have the support of the report in making the experiment of developing this sort of work with girls more fully than has been done in the past.

The Chairman remarked that the last words in the footnote referred to by Miss Punnett applied only to Army candidates. (Laughter.)

Mr. Garstang: A question has been raised about the introduction of square root, which is rather an historical point. Dr. Nunn has written a long paper on the subject. In a book* dealing with the teaching of mathematics, by Prof. Smith, a very able professor in matters of education in America, we find that the geometrical method has been given up in favour of the algebraical. I merely point that out to Dr. Nunn, in case he has not noticed it. Obviously his remarks implied that the earlier method was the geometrical one, of which Dr. Nunn has indicated the origin. But I have read Dr. Nunn's paper, and I confess that I am not convinced by it. I think it is a very able paper; but I am not assured that it gives the proper way, or the only way, in which to teach boys square root. I think it would be a mistake to let this particular paper on square root receive official sanction, even indirectly. It would lead to curtailment in teaching. I do not object to square root being cut out of examinations; but it seems to me that easy questions on square root give the means of introducing a certain amount of algebraic form in a tentative way; and that it is just in that way the best progress is made. The

* *The Teaching of Elementary Mathematics*, by Prof. David Eugene Smith, pp. 131-133.

basis of the algebraic method is the use of the simplest identity; and knowledge of this may be obtained naturally if pupils are encouraged to use notebooks for recording the results of their work.

It is most important that we should preserve the freedom of the teacher. The report as issued advocates alternatives. If the examining boards can be induced to set these alternative questions, we should, after several years' experience of the various methods, be able possibly to reach a conclusion as to which of them were superior and which inferior. At present, I think, we can hardly judge.

Incidentally the last speaker mentioned the various differences between boys and girls. It has been my pleasure to teach both boys and girls in the same class for many years; and, although I have perhaps only a limited experience of girls, I should be inclined to say that they are less interested in mathematics than are boys. I think this is largely because boys are more interested in mechanics and physics; but some girls I have taught have been more interested than the average of the boys. I have found that these special girls are particularly interested in the pure aspect of mathematics, and it has been a revelation to me to hear that girls are more interested in mathematics with a practical application. It is because this report has advocated a kind of practical mathematics that I have ventured to ask one or two ladies on the Committee to put forward their view. I do not think that I can speak with authority on the subject of teaching girls, but I do think that any proposed change in mathematics that would give less encouragement for girls to take part in the pure elements would be to their disadvantage. I have in my mind now the work of a lady, Mrs. Alice Stott, on the Pure Geometry of the Fourth Dimension. I may be wrong, but it seems to me that where you get original mathematics with ladies, it is usually of the pure mathematical kind and not of the applied kind. I am not sure whether it would not be a wise step on the part of the Association to ask those ladies who are on the Committee to act as a special sub-committee in the interests of girls, and to say whether we have, in this report, done wisely or unwisely in regard to the teaching of girls.

EVENING MEETING.

MEMBERS of the Mathematical Association and of the Association of Public School Science Masters dined together at the Criterion Restaurant on Wednesday evening, the 11th of January, Professor H. H. Turner, F.R.S., the retiring president of the Mathematical Association, occupying the chair. Among the most distinguished guests were the Reverend Canon Wilson,

D.D.; Professor H. T. Bovey, F.R.S.; and Professor E. W. Hobson, F.R.S. Dinner was followed by an interesting programme of music and recitations, particulars of which are printed below.

On this occasion an exception was made to our usual practice of omitting toasts and speeches. The toast of Canon Wilson was drunk with enthusiasm, and he was prevailed upon to give some interesting reminiscences. After explaining how he came to be one of the founders of the Association for the Improvement of Geometrical Teaching, which subsequently developed into the Mathematical Association, the speaker told how he was appointed by Dr. Temple to be the first science master at Rugby. At that time he scarcely knew the difference between a thermometer and a barometer, or between oxygen and nitrogen, but he was allowed six months in which to prepare, and after working in the best laboratories available at that time, he commenced his teaching at Rugby, picking up the work with his boys as he went along and never pretending omniscience. Under his supervision the first science laboratory at Rugby was constructed, with accommodation for seven pupils working at one time.

Professor Turner has very kindly allowed us to reprint the words of his song on Halley's comet. It originally appeared in the *Observatory* for March, 1910, and was sung, with an additional (fourth) verse, at the R.A.S. Club dinner, June 10th, 1910, when Drs. Cowell and Crommelin were Club guests. It will be noticed, therefore, that the personal allusions are for home consumption only.

HALLEY'S COMET.

Air—"Sally in our Alley."*

Of all the meteors in the sky
  There's none like Comet Halley.
We see it with the naked eye
  And periodically.
The first to see it was not he,
  But still we call it Halley;
The notion that it would return
  Was his originally.

Of all the years we've lately seen
  There's none to rival this year,
Because though busy we have been
  We're likely to be busier.
When five and twenty years are out
  We look for Comet Halley:
He said it would come back again,
  Perhaps perpetually.

We probe the secrets of the sun,
  And most effectually;
There's much good honest work been done
  Selenographically.
Whatever quest may prove the best
  We all revere bold Halley,
Who said his comet would return,
  And mathematically.

They say the greatest man is not
  A hero to his valet;
But I'd rejoice if 'twere my lot
  To serve the fame of Halley.
So Cowell and Crommelin, Davidson,
  Knox-Shaw, and all the tally,
We'll toast the splendid work they've done
  Enthusiastically.

H. H. T.

* For *Halleius Noster* (=Our 'Alley) see *Principia*, lib. iii. prop. xx. *et passim*.

PROGRAMME.

Song,	"Halley's Comet,"	Professor H. H. Turner.
Song,	"The Heavy Dragoon,"	Mr. M. I. Trachtenberg.
Recitation,	"Painters and Pictures,"	Mr. G. H. Martin.
Song,	"The Gentle Maiden,"	Mr. P. Lavender.
Violin solo,	"Benedictus,"	Mr. E. H. Butt.
Song,	"Loyal Death,"	Mr. W. J. Dobbs.
Recitation,		Mr. A. H. Spiers.
Song,	"The Three-Foot Rule,"	Professor H. H. Turner.
Song,	"Aus der Weser,"	Mr. P. Lavender.
Song,	"The Modern Major-General,"	Mr. M. I. Trachtenberg.
Violin solo,	"Cavatina,"	Mr. E. H. Butt.
Song,	"A Capital Ship,"	Mr. A. H. Spiers.
Recitation,	"A, B, and C,"	Mr. G. H. Martin.

EXHIBIT OF MODELS, ETC.

During the Annual Meeting an exhibition of mathematical models, instruments and books was held on the fourth floor of the building. The following is a list of exhibits and exhibitors:

Educational models, Mr. P. Abbott.

Graphic Arithmetic Charts, Mr. H. D. Ellis.

Models in Mechanics, Mr. G. Goodwill.

The Willis Planimeter, stereoscopic models of catenaries and gyroscopic curves, Sir George Greenhill, F.R.S.

Dr. Hutchinson's Protractor, for use in constructing stereographic and gnomonic projections of the sphere, also slide rules, Messrs. Harling & Co.

Model of the Triple Generation of Three Bar Motion Curve, Colonel R. L. Hippisley, R.E., C.B.

Sewing curves, Miss Home.

Folding models to illustrate the formulae of right-angled spherical triangles, and various graphic solutions, Mr. E. M. Langley.

Calculating machines, Messrs. C. E. Layton.

Models illustrating Fourier's Series, the compounding of harmonic curves, Wallis' 'Arithmetic of Infinites,' also experiments in mechanics, and diagrams illustrating the teaching of algebra, Dr. T. Percy Nunn.

Micrometers, Mr. Ciceri Smith.

Cylindrical and other slide rules, and surveying instruments, Messrs. Stanley & Co.

The Royal Observatory's model of the Jovian system, also a Peaucellier Cell to illustrate the law of the inverse square of the distance, Professor H. H. Turner, F.R.S.

Collections of Mathematical books were shown by Messrs. Edward Arnold, Messrs. George Bell & Sons, Messrs. Blackie, the Cambridge University Press (Mr. C. F. Clay); Messrs. Clive & Sons, Messrs. Dent & Co., Messrs. Ginn & Co., Messrs. Longmans & Co., Messrs. Macmillan & Co., Mr. John Murray, and the Oxford University Press (Messrs. Frowde & Co).

GEOGRAPHICAL DISTRIBUTION OF MEMBERS.

A list of members of the Association on January 1st, 1911, classified according to locality, shows a total of 616, of whom England contains 490, Wales 22, Scotland 22, Ireland 12, Isle of Man 1, Channel Islands 2, South Africa 20, Canada 1, Australia 6, New Zealand 4, West Indies 1, Malta 1, India 13 (viz. Bengal 1, Bombay 1, Madras 7, North-West 3, Burma 1), Brazil 1, Egypt 2, Italy 1, Japan 2, Roumania 1, Syria 1, U.S.A. 13.

The 490 members in England are distributed thus: London 133, Bedfordshire 5, Berkshire 13, Bucks 12, Cambridge 30, Cheshire 6, Cornwall 1, Cumberland 4, Derbyshire 4, Devonshire 10, Dorset 1, Durham 5, Essex 10, Glos. 24, Hants 18, Herts 12, Hereford 2, Isle of Wight 3, Kent 22, Lancashire 24, Leicester 4, Lincs 1, Middlesex 8, Monmouth 1, Norfolk 5, Northants 3, Northumberland 3, Notts 3, Oxford 13, Rutland 4, Shropshire 4, Somerset 6, Staffs 3, Suffolk 1, Surrey 22, Sussex 14, Warwickshire 11, Westmorland 1, Wilts 1, Worcs. 8, Yorkshire 35.

CORRESPONDENCE.

HARROW SCHOOL, Jan. 20th, 1911.

To the Editor of the Mathematical Gazette.

DEAR SIR,

Canon Wilson,* a reformer of forty years' standing, has kindly sent me the enclosed letter with a view to its suggestions being laid before the M.A. Teaching Committee. The letter seems to me of such general interest that I have obtained leave from Canon Wilson to send it to you for publication in the *Gazette*.

Trusting that you will be able to find space for it.

I am, Yours truly, A. W. SIDDONS.

COLLEGE, WORCESTER, Jan. 17th, 1911.

DEAR MR. SIDDONS,

You were good enough to say that I might commit to writing some suggestions that occurred to me in reading the Draft Report of your "Committee on the Teaching of Algebra and Trigonometry," and that you would lay them before the Committee.

My conviction that you have a great opportunity before you, and my strong desire that you should use it to the full, make me venture to address you. I think that what I suggest may give greater unity to your report as a whole, bringing out more clearly a central principle, and showing the relation to it of your analysis of the aims of teaching algebra in § II., and of your recommendations.

I think that you should make it clear *ab initio* that it is a *teaching syllabus for schools* you are outlining, and not a *philosophic basis for algebra*. Few men, and still fewer boys and girls, are capable of what you describe as "scientific interest in form" in the examination of fundamental laws. Historically and educationally algebra is generalised arithmetic.

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* The Ven. J. M. Wilson, D.D., was Senior Wrangler in 1859, and the next year was elected to a Fellowship at St. John's Coll., Cam. For twenty years he was Mathematical and Science Master at Rugby. His eleven years as Head Master of Clifton College marked a period of great educational activity, during which his interest in the Association for the Improvement of Geometrical Teaching continued unabated. His *Elementary Geometry* passed through many editions and reprints. This was an able attempt to deal with a problem that cannot yet be said to have received an entirely satisfactory solution. With the "Syllabus of the Association" it received an amusing castigation at the hands of that doughty defender of undiluted Euclid—C. L. Dodgson, better known, perhaps, to the younger generation as the author of *Alice in Wonderland* (*v.* "Euclid and his Modern Rivals," Act II. scene vi. and Act III. scene ii., in which Nostradamus—*nostra*, pl. of *nostrum*, 'a quack remedy,' and *damus*, 'we give'—may possibly be intended for Dr. Wilson himself). It is almost an impertinence to add that the re-appearance in our midst at the annual meeting of one of whom it might almost be said of his connection with the work of the Association—*quorum pars magna fuit*—has been to many of us not merely an awakening of memories of the early labours of nearly half a century ago, but a potent stimulus to renewed effort. [Ed.]

The algebra, therefore, that you here speak of is the algebra for school, arithmetical in origin, practical in its early applications. There is much which is of interest to pure mathematicians, which is an after-thought in the history of algebra, and is out of place in teaching the young.

When a boy is for the first time introduced to algebra, he wishes, or ought to wish, and may be induced to wish, to know what is the use of it; and it is very desirable to gratify his wish, and not only to tell him that he will know some day. This then appears to me to be the right moment for asking him, and helping him to think out, what mathematics are. How would he define mathematics and their use? Then the teacher criticises, corrects, enlarges, retrenches, etc., the boy's proposed definition, in Socratic method, till he finally leads him to discover for himself Comte's definition, *mathematics is the science of indirect measurement of magnitude, and the processes subsidiary thereto.*

That is the best definition of algebra for boys.

Every word in the above definition must be examined and shown to be important,—*science*, *indirect measurement*, *magnitude*,—and thoroughly illustrated. The boy will soon see that there must be a unit—a foot, a ton, a velocity, and so on—in any subject to which mathematics can be applied, and illustrations will be readily found. Physical science furnishes data and relations between them by direct measurement; and mathematics enables us to measure things indirectly which we cannot measure directly. It gives a boy a new sense of the meaning of geometry, if he is taught that every theorem can be represented and used as an indirect measurement of some magnitude; that Euclid i. 4 enables him to measure the distance across a pond or through a haystack, and Euclid i. 26, with a little extension to proportion, shows him how to measure indirectly the distance of an inaccessible object, a buoy at sea, the moon, even the fixed stars. He will believe in the use of geometry as a science of indirect measurement.

But algebra? Of what use is it? I should tell him at once that algebra also is employed for the indirect measurement of magnitudes of all sorts, and that this measurement is generally effected by the *formation and solution of equations* which involve the quantity to be measured. This measurement may be of such varied things as the weight of the sun or moon, the velocity of light, the thickness of a soap-bubble, the horse-power of an engine, the temperature of a furnace, or the mass and velocity of an electron, all incapable of direct measurement. This he can only understand as a generalization from some examples. Take him at once into a problem. "A fish is caught, too heavy for the weights on our scale; it is cut into three parts, head, tail and body; and it is noticed that the body weighs as much as the head and tail; that the head weighs as much as the tail and half the body; and that the tail weighs 9 lbs. What is the weight of the fish?" Can we indirectly measure it?

The boy is led slowly and thoroughly to make his equation, using all the data. Then follows the axioms,—if equals be added to equals, etc., and the equation is solved; and the solution verified. *He has learnt the use of algebra* as a *science of indirect measurement.*

It is quite possible, and highly stimulating, to introduce here problems which lead to simultaneous, or quadratic, or cubic equations, and to leave these equations formed, but unsolved: and show the need of "subsidiary processes."

This gives a boy a map of the whole field he has to traverse, and a clue to his path; it satisfies his natural desire to see the use of what he is doing: it gives him patience in working at the "subsidiary processes."

But it also serves another purpose. It gives a valuable guide both to the order of teaching, and to the extent to which each subsidiary process should be studied and developed. Subsidiary processes should be carried, in the first instance, just so far as they are really subsidiary and no further, not as

mere curious exercises. They are subsidiary to the two purposes, of solving equations, and of otherwise effecting indirect measurements.

The latter purpose justifies the retention, at an early stage, of some other subjects besides equations, such as progressions. A boy is told to add $1+2+3 \ldots$ up to 100. The direct method is of course possible, but tedious; teach him to take them in pairs $1+100$, $2+99$, $3+98$, and he sees that there are 50 pairs of 101 each, and indirectly gets the amount. So the principle justifies the retention of permutations and combinations as indirect ways of counting. Tell a boy to draw a regular decagon and all its diagonals; and then to count how many lines he has drawn. He will succeed probably in counting 45. Now tell him to produce every line in both directions, and then count how many triangles there are on the paper; and that in a fortnight he will be able to say there are exactly 10,000. The use of logarithms is justified to calculate indirectly such results as the number of digits in 2^{100}. Tell him the old story of the regiment of 100 monkeys, the first of whom came to a pile of nuts, and took half the pile and one more; the second took half of what was left and one more; and so on, till the last monkey did the same, and finished the pile. How many nuts were there? And how large a cube would they fill if the diameter of a nut was one-third of an inch?

I need not go into the application of the same principle to trigonometry. The indirect measurement of distances, areas, angles, etc., in teaching this subject takes the same place as it does in algebra, and similarly depends on the formation of equations, and on solving them by means of subsidiary processes.

If this principle is grasped in your report, and recommended for adoption in the construction of syllabuses and text books, it seems to me that more rapid progress will be made by boys into the heart and application of mathematical subjects: they would sooner be ready for mechanics and the calculus; and would gain much from beginning them; and less time will be wasted, or nearly wasted, in teasing boys with what seem to them useless manipulations, clever dodges. The ingenious proofs of identities in algebra and trigonometry, for example, will find a subordinate place outside the course which the whole stream of boys will follow.

The boy who is capable of becoming a philosophic mathematician will not be sacrificed by this order of teaching: and the mass of boys will take more interest in mathematics, and learn far more of the power that mathematical methods give.

If I may make a general remark on the teaching of science I should say that we are in danger of sacrificing the natural *educational* method to what appears a *logical* method. In teaching botany it is an educational blunder to begin with the cell: one should begin with the flower and comparison of flowers and pass to generalizations, ever widening: so botany developed. In mechanics it is an educational blunder to begin with the vector: one begins with the lever. All science teaching should proceed from the known to the unknown, and the history of its development is a good guide. We must avoid this mistake in teaching mathematics.

I have not attempted to work this suggestion into your report or to examine how far your report would need to be modified. This must be for you to consider. But I trust that you will find some truth and value in my suggestions.

Very truly yours, JAMES M. WILSON.

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